CS717 Spring 06
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Comments to Assignments #5 and #6
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II.31 if x has no Cauchy subsequence, then all Cauchy sequences in $X := \{x_n : n \in \mathbb{N}\}$ converge (indeed, if $y : \mathbb{N} \to X : n \mapsto y_n$ is Cauchy, then, with $m : \mathbb{N} \to \mathbb{N} : n \mapsto \min\{r : y_n = x_r\}$, we cannot have $\sup_n m(n) = \infty$ since then there would be a strictly increasing $p : \mathbb{N} \to \mathbb{N}$ so that $m \circ p$ is strictly increasing (e.g., for all n choose $p(n+1)$ to be the smallest r for which $m(r) > p(n)$), hence $n \mapsto y_{p(n)} = x_{m(p(n))}$ would be a subsequence of both y and x, hence both a Cauchy sequence and not a Cauchy sequence; thus m is bounded, therefore $\operatorname{ran} y$ is finite, hence the nonincreasing sequence $n \mapsto \operatorname{diam} y_{>n}$ can have at most finitely many jumps, and, as it converges to zero, it must be identically zero from some point on, i.e., y must be eventually constant, hence converges trivially), i.e., X is complete, hence compact, hence (x_n) has convergent subsequences after all.

The standard proof: Construct that Cauchy subsequence as follows: Since X is totally bounded, it can be covered by finitely many balls of radius r, whatever $r > 0$ might be, hence, any subsequence of it, since it has *infinitely* many entries, has a subsequence that lies entirely in some ball of radius r. Apply this as follows: With x^0 the original sequence, pick, for $k = 1, 2, 3, \ldots$, a strictly increasing $p^k : \mathbb{N} \to \mathbb{N}$ so that $x^k : n \mapsto x^{n-1}_{p^k(n)}$ has all its terms in a ball of radius $1/k$. Then $y : n \mapsto x^n_n = x_{p^n(p^{n-1}\cdots(p^1(n))\cdots)}$ is a subsequence of x with $\operatorname{diam} y_{\ge n} \le 2/n$ for all n, hence y is Cauchy.

III.4 (i) $x \mapsto d(x, Y)$ is nonnegative. (ii) $d(0x, Y) = d(0, Y) = 0 = 0d(x, Y)$ since $0 \in Y$; also, for $\alpha \neq 0$, $Y/\alpha = Y$, hence $d(\alpha x, Y) = d(\alpha x, \alpha Y/\alpha) = d(\alpha x, \alpha Y) = |\alpha| d(x, Y)$, by scale-invariance of norm metric, thus $x \mapsto d(x, Y)$ is positive homogeneous. (iii) $d(x + z, Y) = \inf_{y,y' \in Y} \|x+z-y-y'\| \le \inf_{y,y' \in Y} \|x-y\|+\|z-y'\| = \inf_{y \in Y} \|x-y\|+\inf_{y' \in Y} \|z-y'\| = d(x, Y) + d(z, Y)$ (the second-last equality since y, y' range independently over Y), hence $x \mapsto d(x, Y)$ is subadditive.

By H.P.1, X/Y is nls with respect to $\langle x \rangle \mapsto d(x, Y)$ in case $Y = \ker d(\cdot, Y)$, i.e., in case $Y = Y^-$. Conversely, if $\langle x \rangle \mapsto d(x, Y)$ is a norm, then, in particular, $x \in Y^-$ implies $d(x, Y) = 0$ implies $\|\langle x \rangle\| = 0$ implies $\langle x \rangle = 0$, i.e., $x \in Y$, i.e., Y is closed.

III.10 If $A \in bL(X, Y)$ is bounded below and onto, then it is 1-1 and onto, hence invertible, and, as mentioned in the notes, the assumed lower bound provides a bound for A^{-1}, i.e., A^{-1} is bounded, hence continuous, hence, by (II.7), its inverse, i.e., A, carries open sets to open sets.

III.11 Since $Y = \operatorname{tar} A$ is ms, sufficient to prove that $\operatorname{ran} A$ is sequentially closed, i.e., that $y = \lim_{n\to\infty} Ax_n$ implies that $y \in \operatorname{ran} A$. For this, since $c := \inf_x \|Ax\|/\|x\| > 0$, have $\operatorname{diam}(x_{>n}) \le (1/c) \operatorname{diam}(Ax_{>n}) \to 0$ as $n \to \infty$ since (Ax_n), being convergent, is Cauchy. Since X is complete, this implies that $x := \lim_{n\to\infty} x_n$ is a well-defined element of X, and, by the continuity of A, $Ax = \lim_{n\to\infty} Ax_n = y$. In particular, $y \in \operatorname{ran} A$.

III.13 Since Y is closed lss of X, H.P.(4) states that X/Y is nls with respect to the norm $\|\langle x \rangle\| := d(x, Y)$, and, with respect to this norm, the quotient map $\langle\rangle : X \to X/Y$ is continuous (since it is linear and bounded, in fact $\|\langle\rangle\| \le 1$). Since $Y + Z$ is the inverse image, under this continuous map, of the set $\langle\rangle(Z)$, and this set is a finite-dimensional lss of $X/Y = \operatorname{tar}\langle\rangle$, hence closed (by H.P.(12)), $Y + Z$ itself is closed.

III.17 $A \neq 0$ implies that $X \neq \ker A$. Hence $\|A\| = \sup_{x\in X} \frac{\|Ax\|}{\|x\|} = \sup_{x\in X} \sup_{y\in\ker A} \frac{\|A(x-y)\|}{\|x-y\|} = \sup_{x\in X} \frac{\|Ax\|}{\inf_{y\in\ker A}\|x-y\|} = \sup_{x\in X} \frac{\|Ax\|}{d(x,\ker A)} = \sup_{x\in X} \frac{\|A_|\langle x\rangle\|}{\|\langle x\rangle\|} = \|A_|\|$.

III.20 Since $\lim_{n\to\infty} A_n = A$, there is m so that $A_{>m}$ is in $B_r(A)$ with $r := 1/\|A^{-1}\|$, hence A_n is boundedly invertible for all $n > m$, by (17)Proposition, with $A_n^{-1} = (A^{-1}A_n)^{-1}A^{-1}$, and $\|(A^{-1}A_n)^{-1}\| \le 1/(1 - \|A^{-1}(A - A_n)\|) \to 1$ as $n \to \infty$, hence $\limsup_n \|A_n^{-1}\| \le \|A^{-1}\|$. In particular, A_n^{-1} is bounded uniformly in n for large n.

$A^{-1} - Q^{-1} = A^{-1}(Q - A)Q^{-1}$ (multiply out), hence $\|A - Q\| \to 0$ implies $\|Q^{-1}\|$ is eventually bounded, and therefore also $\|A^{-1} - Q^{-1}\| \to 0$.

IV.1 If P denotes the map in question, then $P = [y/\lambda y]\circ\lambda$, while $\lambda[y/\lambda y] = 1$. Hence P is the linear projector with $\operatorname{ran} P = \operatorname{ran}[y]$ and $\operatorname{ran} P' = \operatorname{ran}[\lambda]$, i.e., its interpolation functionals are of the form $\alpha\lambda$ for $\alpha \in \mathbb{F}$.

IV.2 The inequality is a consequence of the definition of $\|\lambda\|$.

$H(\lambda, \lambda k) = k + \ker\lambda$, and $B_r(x) - k = B_r(x - k)$, and, for any two subsets M and N of X, $M \cap (N + k) = k + (M - k) \cap N$. Therefore, with $y := x - k$, the given condition is equivalent to

(i) $B^-_{\|y\|}(y) \cap \ker\lambda \neq \{\} = B_{\|y\|}(y) \cap \ker\lambda$.

This condition implies that, for some $u \in \ker\lambda$, $\|y - u\| = \|y\|$ while, for all $z \in \ker\lambda$, $\|y - z\| \ge \|y\|$, hence $\inf_{z\in\ker\lambda}\|y - z\| = \|y\|$, therefore (since $0 \in \ker\lambda$)

(ii) 0 is a ba to y from $\ker\lambda$.

This, in turn, implies (i) (since it says that $0 \in B^-_{\|y\|}(y) \cap \ker\lambda$ and that $\|y - z\| \ge \|y\|$ for all $z \in \ker\lambda$). Finally, (ii) is equivalent to $|\lambda(y)| = \|\lambda\|\|y\|$, by (12)Corollary.

IV.6 $|\sum_{u\in U} a(u)f(u)| \le \sum_{u\in U} |a(u)||f(u)| \le \|a\|_1\|f\|_\infty$ hence $\|\sum_{u\in U} a(u)\delta_u\| \le \|a\|_1$ regardless of whether or not U is finite.

For finite U, get equality by choosing $f = \sum_{u\in U} \operatorname{signum} a(u) l_u$, with $l_p := t \mapsto (1 - d(t, p)/s)_+$ (which is continuous as the composition of continuous maps ($t \mapsto d(t, p)$, $t \mapsto 1 - t/s$, $t \mapsto t_+$) for positive s , with s chosen so that $0 < s < \min\{d(u, w)/2 : u, w \in U, w \neq u\}$ (that minimum is indeed positive since U is finite), therefore $\|f\|_\infty = 1$.

For nonfinite U , $\sum_{u\in U} a(u)\delta_u$ is the norm-limit of $\sum_{u\in W} a(u)\delta_u$ with W finite subsets of U (since $\|a\|_1 < \infty$), hence its norm is still $\|a\|_1$.