II.31 if \( x \) has no Cauchy subsequence, then all Cauchy sequences in \( X := \{ x_n : n \in \mathbb{N} \} \) converge (indeed, if \( y : \mathbb{N} \to X : n \mapsto y_n \) is Cauchy, then, with \( m : \mathbb{N} \to \mathbb{N} : n \mapsto \min \{ r : y_n = x_r \} \), we cannot have \( \sup_n m(n) = \infty \) since then there would be a strictly increasing \( p : \mathbb{N} \to \mathbb{N} \) so that \( m \circ p \) is strictly increasing (e.g., for all \( n \) choose \( p(n+1) \) to be the smallest \( r \) for which \( m(r) > p(n) \)), hence \( n \mapsto y_{p(n)} = x_{m(p(n))} \) would be a subsequence of both \( y \) and \( x \), hence both a Cauchy sequence and not a Cauchy sequence; thus \( m \) is bounded, therefore \( \text{ran} \ y \) is finite, hence the nonincreasing sequence \( n \mapsto \text{diam} \ y_{\geq n} \) can have at most finitely many jumps, and, as it converges to zero, it must be identically zero from some point on, i.e., \( y \) must be eventually constant, hence converges trivially), i.e., \( X \) is complete, hence compact, hence \( (x_n) \) has convergent subsequences after all.

The standard proof: Construct that Cauchy subsequence as follows: Since \( X \) is totally bounded, it can be covered by finitely many balls of radius \( r \), whatever \( r > 0 \) might be, hence, any subsequence of it, since it has \( \text{infinitely} \) many entries, has a subsequence that lies entirely in some ball of radius \( r \). Apply this as follows: With \( x^0 \) the original sequence, pick, for \( k = 1, 2, 3, \ldots \), a strictly increasing \( p^k : \mathbb{N} \to \mathbb{N} \) so that \( x^{n_k} = x_{p^k(n)} \) has all its terms in a ball of radius \( 1/k \). Then \( n \mapsto x^n = x_{p^n(p^{n-1}(\ldots p^1(n))\ldots)} \) is a subsequence of \( x \) with \( \text{diam} \ y_{\geq n} \leq 2/n \) for all \( n \), hence \( y \) is Cauchy.

III.4 (i) \( x \mapsto d(x, Y) \) is nonnegative. (ii) \( d(0x, Y) = d(0, Y) = 0 = 0d(x, Y) \) since \( 0 \in Y \); also, for \( \alpha \neq 0 \), \( Y/\alpha = Y \), hence \( d(\alpha x, Y) = d(\alpha x, \alpha Y)/\alpha = d(\alpha x, \alpha Y) = |\alpha|d(x, Y) \), by scale-invariance of norm metric, thus \( x \mapsto d(x, Y) \) is positive homogeneous. (iii) \( d(x + z, Y) = \inf_{y,y' \in Y} \| x + z - y - y' \| \leq \inf_{y,y' \in Y} \| x - y \| + \| z - y' \| = \inf_{y \in Y} \| x - y \| + \inf_{y' \in Y} \| z - y' \| = d(x, Y) + d(z, Y) \) (the second-last equality since \( y, y' \) range independently over \( Y \)), hence \( x \mapsto d(x, Y) \) is subadditive.

By H.P.1, \( X/Y \) is nls with respect to \( \langle x \rangle \mapsto d(x, Y) \) in case \( Y = \ker d(\cdot, Y) \), i.e., in case \( Y = Y^- \). Conversely, if \( \langle x \rangle \mapsto d(x, Y) \) is a norm, then, in particular, \( x \in Y^- \) implies \( d(x, Y) = 0 \) implies \( \| \langle x \rangle \| = 0 \) implies \( \langle x \rangle = 0 \), i.e., \( x \in Y \), i.e., \( Y \) is closed.

III.10 If \( A \in bL(X, Y) \) is bounded below and onto, then it is 1-1 and onto, hence invertible, and, as mentioned in the notes, the assumed lower bound provides a bound for \( A^{-1} \), i.e., \( A^{-1} \) is bounded, hence continuous, hence, by (II.7), its inverse, i.e., \( A \), carries open sets to open sets.

III.11 Since \( Y = \text{tar} \ A \) is ms, sufficient to prove that \( \text{ran} \ A \) is sequentially closed, i.e., that \( y = \lim_{n \to \infty} Ax_n \) implies that \( y \in \text{ran} \ A \). For this, since \( c := \inf_{x} \| Ax \|/\| x \| > 0 \), have \( \text{diam}(x_{>n}) \leq (1/c) \text{diam}(Ax_{>n}) \to 0 \) as \( n \to \infty \) since \( (Ax_n) \), being convergent, is Cauchy. Since \( X \) is complete, this implies that \( x := \lim_{n \to \infty} x_n \) is a well-defined element of \( X \), and, by the continuity of \( A \), \( Ax = \lim_{n \to \infty} Ax_n = y \). In particular, \( y \in \text{ran} \ A \).

III.13 Since \( Y \) is closed lss of \( X \), H.P.(4) states that \( X/Y \) is nls with respect to the norm \( \| \langle x \rangle \| := d(x, Y) \), and, with respect to this norm, the quotient map \( \langle \cdot \rangle : X \to X/Y \) is continuous (since it is linear and bounded, in fact \( \| \langle \cdot \rangle \| \leq 1 \)). Since \( Y + Z \) is the inverse
image, under this continuous map, of the set \( \langle (Z) \rangle \), and this set is a finite-dimensional lss of \( X/Y = \text{tar} \langle \rangle \), hence closed (by H.P. (12)), \( Y + Z \) itself is closed.

**III.17** \( A \neq 0 \) implies that \( X \neq \text{ker} A \). Hence \( \| A \| = \sup_{x \in X} \frac{\| Ax \|}{\| x \|} = \sup_{x \in X} \frac{\| A(x-y) \|}{\| x-y \|} = \sup_{x \in X} \frac{\| A \|}{\inf_{y \in \text{ker} A} \| x-y \|} = \sup_{x \in X} \frac{\| A \|}{d(x, \text{ker} A)} = \sup_{x \in X} \frac{\| A \|}{\| x \|} = \| A \|.

**III.20** Since \( \lim_{n \to \infty} A_n = A \), there is \( m \) so that \( A_{m-1} \) is in \( B_r(A) \) with \( r := 1/\| A^{-1} \| \), hence \( A_n \) is boundedly invertible for all \( n > m \), by (17)Proposition, with \( A_n^{-1} = (A^{-1} A_n)^{-1} A^{-1} \), and \( \| (A^{-1} A_n)^{-1} \| \leq 1/(1 - \| A^{-1} (A - A_n) \|) \to 1 \) as \( n \to \infty \), hence \( \lim \sup_n \| A_n^{-1} \| \leq \| A^{-1} \| \). In particular, \( A_n^{-1} \) is bounded uniformly in \( n \) for large \( n \).

\( A - Q^{-1} = A^{-1} (Q - A) Q^{-1} \) (multiply out), hence \( \| A - Q \| \to 0 \) implies \( \| Q^{-1} \| \) is eventually bounded, and therefore also \( \| A - Q^{-1} \| \to 0 \).

**IV.1** If \( P \) denotes the map in question, then \( P = [y/\lambda y] \circ \lambda \), while \( \lambda [y/\lambda y] = 1 \). Hence \( P \) is the linear projector with \( \text{ran} P = \text{ran} [y] \) and \( \text{ran} P' = \text{ran} [\lambda] \), i.e., its interpolation functionals are of the form \( \alpha \lambda \) for \( \alpha \in \mathbb{F} \).

**IV.2** The inequality is a consequence of the definition of \( \| \lambda \| \).

\( H(\lambda, \lambda k) = k + \ker \lambda \), and \( B_r(x) = B_r(x-k) \), and, for any two subsets \( M \) and \( N \) of \( X \), \( M \cap (N+k) = k + (M-k) \cap N \). Therefore, with \( y := x-k \), the given condition is equivalent to

(i) \( B_{\| y \|} (y) \cap \ker \lambda \neq \{ \} = B_{\| y \|} (y) \cap \ker \lambda \).

This condition implies that, for some \( u \in \ker \lambda \), \( \| y-u \| = \| y \| \) while, for all \( z \in \ker \lambda \), \( \| y-z \| \geq \| y \| \), hence \( \inf_{z \in \ker \lambda} \| y-z \| = \| y \| \), therefore (since \( 0 \in \ker \lambda \))

(ii) \( 0 \) is a ba to \( y \) from \( \ker \lambda \).

This, in turn, implies (i) (since it says that \( 0 \in B_{\| y \|} (y) \cap \ker \lambda \) and that \( \| y-z \| \geq \| y \| \) for all \( z \in \ker \lambda \)). Finally, (ii) is equivalent to \( |\lambda(y)| = \| \lambda \| \| y \| \), by (12)Corollary.

**IV.6** \( |\sum_{u \in U} a(u) f(u)\| \leq \sum_{u \in U} |a(u)| \| f(u)\| \leq \| a \|_1 \| f \|_\infty \) hence \( \| \sum_{u \in U} a(u) f(u) \| \leq \| a \|_1 \) regardless of whether or not \( U \) is finite.

For finite \( U \), get equality by choosing \( f = \sum_{u \in U} \text{signum} \ a(u) l_u \), with \( l_p := t \mapsto (1-d(t,p)/s)_+ \) (which is continuous as the composition of continuous maps \( t \mapsto d(t,p), t \mapsto 1-t/s, t \mapsto t_+ \) for positive \( s \), with \( s \) chosen so that \( 0 < s < \min\{d(u,w)/2 : u, w \in U, w \neq u \} \) (that minimum is indeed positive since \( U \) is finite), therefore \( \| f \|_\infty = 1 \).

For nonfinite \( U \), \( \sum_{u \in U} a(u) f(u) \) is the norm-limit of \( \sum_{u \in W} a(u) f(u) \) with \( W \) finite subsets of \( U \) (since \( \| a \|_1 < \infty \)), hence its norm is still \( \| a \|_1 \).

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