CS717 Spring 06
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Comments to Assignments #7 and #8
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IV.4 (i) ν is cont. sublinear fl, hence its kernel, i.e., c_0, is closed.

(ii) $\lim_{n\to\infty}\|x-P_nx\| = \lim_{n\to\infty}\sup_{j\ge n}|x(j)| = \limsup_{n\to\infty}|x(n)| = 0$ for $x\in c_0$, by (i).

(iii) Using the continuity of $\lambda\in(c_0)^*$, $\lambda x = \lim_{n\to\infty}\lambda(P_nx)$
$= \lim_{n\to\infty}\sum_{j\le n}\lambda(e_j)x(j) = \lim_{n\to\infty}\sum_{j\le n}y_\lambda(j)x(j) = \sum_{j=1}^\infty y_\lambda(j)x(j)$.

(iv) By (iii) and Hölder, $\|\lambda\|\le\|y_\lambda\|_1$, while $\|\lambda\|\ge\|\lambda P_n\| = \sum_{j\le n}|y_\lambda(j)|\to\|y_\lambda\|_1$ as $n\to\infty$, with the equality also by Hölder.

(v) the map $\iota:\lambda\mapsto y_\lambda$ has been shown to be an isometry from $(c_0)^*$ into ℓ_1. It is linear (since $y_\lambda(j)=\lambda(e_j)$, all j). It is also onto, since, for each $y\in\ell_1$, $\lambda_y : c_0\to\mathbb{F} : x\mapsto\sum_j y(j)x(j)$ is a lfl on c_0 with $\|\lambda_y\|\le\|y\|_1$, and $y_{\lambda_y}=y$. In short, ι is an invertible linear isometry from $(c_0)^*$ to ℓ_1.

IV.10 By the earlier argument, we only have to show the general HB theorem for the special case $X = Y + \mathrm{ran}[z]$.

Any extension g of f to $X := Y+\mathrm{ran}[z]$ is necessarily of the form $g : X\to\mathbb{R} : y+\alpha z\mapsto f(y)+\alpha\zeta$ for some ζ. We want $g\le p$, i.e., that $f(y)+\alpha\zeta\le p(y+\alpha z)$ for all $y\in Y$ and all $\alpha\in\mathbb{R}$. This holds for $\alpha=0$. For $\alpha>0$, can divide both sides by α to get equivalent inequality $f(y/\alpha)+\zeta\le p((y/\alpha)+z)$ to hold for all $y\in Y$, and, since Y is lss, this is equivalent to

$$\zeta\le\inf_{y\in Y}p(y+z)-f(y).$$

For $\alpha<0$, can divide both sides by $-\alpha$ to get equivalent inequality $f(y/(-\alpha))-\zeta\le p(y/(-\alpha)-z)$ to hold for all $y\in Y$, and, since Y is lss, this is equivalent to

$$\sup_{y\in Y}f(y)-p(y-z)\le\zeta.$$

It remains to show that there is such a ζ, but this follows from the fact that, for arbitrary $y,y'\in Y$, $f(y)+f(y') = f(y+y')\le p(y+y') = p((y-z)+(y'+z))\le p(y-z)+p(y'+z)$, or, $f(y)-p(y-z)\le f(y')-p(y'+z)$.

IV.12 By construction, $\lambda:=\delta_{t_0,\dots,t_n}$ is the particular linear combination $\lambda=\sum_{i=0}^n w_i\delta_{t_i}$ of point evaluations at the t_i that satisfies $\lambda\prod_{i=0}^{j-1}(\cdot-t_i)=\delta_{jn}$, $j=0,\dots,n$. In other words, for any $p\in\Pi_n$, λp is the coefficient of $()^n$ in the power form for p. Hence, for $j=0,\dots,n$, $1/\prod_{i\ne j}(t_j-t_i) = \lambda\ell_j = \sum_{i=0}^n w_i\ell_j(t_i) = w_j$, while $\lambda\perp\Pi_{<n}$. Hence, from (33) and with H.P.(7), $d(x,\Pi_{<n})\ge|\lambda x|/\|\lambda\| = |\delta_{t_0,\dots,t_n}x|/\sum_i\prod_{j\ne i}|t_i-t_j|^{-1}$.

IV.16 (a) If $R:=\mathrm{ran}[e_1,e_2,\ldots]$ is not dense in X, then there exists $\lambda\in R^\perp\backslash 0$, hence $\lambda x = c^tx$ implies that $c=0$, i.e., not every $\lambda\in X^*$ can be so represented.

(b) If the norm is monotone, then, for each $x\in X$ and $n\in\mathbb{N}$, $P_nx:=\sum_{j\le n}x(j)e_j$ converges to x (since then $\|x-P_nx\| = d(x,Y_n)$ with $Y_n:=\mathrm{ran}[e_1,\ldots,e_n]$ while, by assumption, $\lim_n d(x,Y_n)=0$. Now use H.P.(4).

IV.19 The f over which we are to maximize $\int_0^1 xf(x)\,\mathrm{d}x =: \nu f$ are all measurable f with $\|f\|_2 = \int_0^1 |f(x)|^2\,\mathrm{d}x = 1$ and $\lambda f := \int_0^1 f(x)\,\mathrm{d}x = 0$. This implies that the maximum in question is the norm of ν, as a lfl on $L_2[0\,..\,1]$, but restricted to $Y := \ker\lambda$. By (39), this norm equals $\min_\alpha \|\nu - \alpha\lambda\|$, while, by Hölder's inequality (with equality case), $\|\nu - \alpha\lambda\| = \|()^1 - \alpha\|_2$, and this is minimum when $\alpha = 1/2$. Thus, the max-value is $\|()^1 - 1/2\|_2 = 1/\sqrt{12}$.

V.2 If $Ax := \lim_n A_n x$ for all $x \in X$, then, by linearity of limits, $A(\alpha x + y) = \lim A_n(\alpha x + y) = \lim \alpha A_n x + A_n y = \alpha \lim A_n x + \lim_n A_n y = \alpha Ax + Ay$, i.e., A is linear. Also, $\|Ax\| = \lim_n \|A_n x\| \le \liminf_n \|A_n\|\|x\|$ for any x, hence $\|A\| \le \liminf_n \|A_n\|$. Hence if $\sup_n \|A_n\| < \infty$, then A is bounded.

V.10 Since any subset of a nowhere dense set is nowhere dense, hence any subset of a thin set is thin, it is sufficient to prove that a totally bounded set in an infinite-dimensional nls is contained in a thin set. If A is totally bounded, then there exists, for each n, a finite $(1/n)$-net V_n for A, therefore $A \subset \{x \in X : \operatorname{dist}(x, \operatorname{ran}[V_n]) \le 1/n\}$, and the latter set is thin, by (4)Proposition, in case X is infinite-dimensional since, in that case, each $\operatorname{ran}[V_n]$, being finite-dimensional, is a proper and closed lss.

V.11 (i) Since V is a Schauder basis by assumption, V^{-1} is well-defined on X. By definition, $P_n = [v_1, v_2, \ldots, v_n]V^{-1}$, hence P_n is a linear map, and maps into $\operatorname{ran}[v_1, v_2, \ldots, v_n]$ and is the identity on that set, hence its range is $\operatorname{ran}[v_1, v_2, \ldots, v_n]$. In particular, $\operatorname{ran} P_n \subset \operatorname{ran}[v_1, v_2, \ldots, v_{n+1}] = \operatorname{ran} P_{n+1}$. Finally, by definition of Schauder basis, $\lim P_n x = x$ for all $x \in X$, i.e., (P_n) converges pointwise to 1.

(ii) For any n, $P_n x - P_{n-1} x = (\lambda_n x)v_n$, hence $|\lambda x| = \|(\lambda x)v_n\| \le (\|P_n\| + \|P_{n-1}\|)\|x\|$, therefore $\sup_n \|\lambda_n\| \le 2\sup_n \|P_n\| < \infty$ by assumption. This implies that $\|V^{-1}x\|_\infty = \sup_n |\lambda_n x| \le \sup_n \|\lambda_n\|\|x\| < \infty$; in particular, $\|V^{-1} : X \to \ell_\infty\| \le 2\sup_n \|P_n\| < \infty$.

V.13 $\mathbf{A}$ pointwise bounded on Z implies that $Z = \cup_n Z \cap C_n$ with $C_n := \{x \in X : \mathbf{A}x \subset B_n^-\} = \cap_{p\in\mathbf{A}} p^{-1}(-\infty\,..\,n]$ closed, hence, since Z is not thin, the closed set $(Z \cap C_n)^- \subseteq Z^- \cap C_n$ has interior for some n, therefore $B_r(x) \subseteq Z^- \cap C_n$ for some $r > 0$, some $x \in Z$ (offhand, $x \in Z^-$, but, by adjusting r, we may take $x \in Z$), and some $n \in \mathbb{N}$. Therefore, $\sup \mathbf{A}(B_r) = \sup \mathbf{A}(B_r(x) - x) \le \sup \mathbf{A}(B_r(x)) + \sup \mathbf{A}(-x) \le 2n =: M$ is finite (using the subadditivity of the elements of $\mathbf{A}$ and the assumed pointwise boundedness of $\mathbf{A}$ on the symmetric Z).

Let $p \in \mathbf{A}$. Then $p(0) \le p(x) + p(-x)$, which, with $x = 0$ implies that $0 \le p(0)$ and so, for general x, if one of $p(x)$ and $p(-x)$ is negative, then its absolute value cannot exceed the value of the other, hence $\max\{p(x), p(-x)\} = \max\{|p(x)|, |p(-x)|\}$ and therefore $\sup |p(B_s)| = \sup p(B_s)$. Also, $p(nx) \le np(x)$ for $n = 1, 2, \ldots$ (though not necessarily for $n = 0$). Therefore, for any $s > 0$, and any $p \in \mathbf{A}$, there is $n \in \mathbb{N}$ so that $nr > s$, hence $\sup |p(B_s)| = \sup p(B_s) \le \sup p(B_{nr}) \le n \sup p(B_r) \le nM < \infty$.