IX. The spectrum of a lm

** point spectrum **

An **eigenpair** for $A \in L(X)$ is any $(x, z) \in (X \backslash 0) \times \mathbb{F}$ satisfying $Ax = zx$. The scalar z in such an eigenpair is called an **eigenvalue** for A and the vector x is called an **eigenvector** for A belonging to the eigenvalue z. The collection of all eigenvectors of A belonging to z is $\ker(A - z) \backslash 0$. Thus z is an eigenvalue of A iff $(A - z)$ fails to be 1-1.

The collection of all eigenvalues of A is called the **point spectrum** of A and is denoted by

$$\sigma_P(A).$$

Eigenstructure of a matrix

If X is finite-dimensional, then A is a matrix and one is naturally led to look into the eigenstructure of A when one looks for a basis V, i.e., an invertible lm $V : \mathbb{F}^n \to X$,

$$\begin{array}{ccc} & A & \\ X & \longrightarrow & X \\ & & \\ V \uparrow & & \uparrow V \\ & & \\ \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ & \hat{A} & \end{array}$$

for which the corresponding **matrix representation** $\hat{A} = V^{-1}AV$ for A is particularly simple. Ideally, one wants $\hat{A}$ to be a diagonal matrix. If there is such V, then A is called **diagona(liza)ble**. Assuming that $\hat{A}$ is diagonal, $\hat{A} = \operatorname{diag}\lceil z_1, \ldots, z_n \rfloor$ say, then necessarily $Av_j = z_j v_j$, all j, i.e., the basis V must consist of (nontrivial) eigenvectors of A.

Whether or not $\hat{A}$ is diagonal, $\hat{A}$ is said to be **similar** to A.

** who cares about eigenstructure? **

If you look into the question as to why one might want a particularly simple matrix representation for A in the first place, you will find that it is useful for understanding the powers of A, of importance in the analysis of fixed-point iteration for solving linear (and nonlinear) systems, the solution of a system of first-order ODEs, and in the numerical solution of evolution equations.

For example, a square matrix A is **powerbounded**, i.e., $\{A^k : k \in \mathbb{N}\}$ is a bounded set iff $\forall\{z \in \sigma_P(A)\}\ |z| \le 1$ with equality only if z is not **defective**, i.e., only if $\operatorname{ran}(A-z) \cap \ker(A - z) = \{0\}$. Further, A is **convergent**, i.e., $\lim_{k\to\infty} A^k$ exists iff $\forall\{z \in \sigma_P(A)\}\ |z| \le 1$ with equality only if z is not defective and $z = 1$. Finally, A is **convergent to** 0, i.e., $\lim_{k\to\infty} A^k = 0$ iff $\forall\{z \in \sigma_P(A)\}\ |z| < 1$ (as was mentioned already in Chapter 2 in the discussion of fixed point iteration).

**** polynomials in a lm ****

More generally, one is interested in understanding the behavior of linear combinations $\sum_{j\le k} a(j)A^j$ of such powers, i.e., of polynomials $p(A)$ in A (with $p := \sum_j ()^j a(j)$), and, ultimately, of functions $f(A)$ in A, to the extent that f can be approximated arbitrarily closely by polynomials p, hence $f(A)$ can be understood as the limit of $p(A)$ as $p \to f$. E.g., $y(t) = \exp(tA)y_0$ is the unique (vector-valued) solution at t of the first-order ODE $Dy = Ay$ with side condition $y(0) = y_0$.

Having a complete understanding of the eigenstructure of A vastly simplifies all dealings with $p(A)$. Indeed, if $A = V\hat{A}V^{-1}$, then, for any $p \in \Pi$,

$$p(A) = Vp(\hat{A})V^{-1},$$

while, for a diagonal matrix $\hat{A} = \operatorname{diag}\lceil \dots, z_j, \dots \rfloor$,

$$p(\hat{A}) = \operatorname{diag}\lceil \dots, p(z_j), \dots \rfloor.$$

Thus, for a diagonalizable A,

$$\sigma_P(p(A)) = p(\sigma_P(A)).$$

This is a particular example of the **Spectral Mapping Theorem**.

Work with polynomials in the lm A is materially helped by the seemingly trivial fact that any two polynomials in the same linear map commute:

$$(1) \qquad \forall\{p, q \in \Pi; A \in L(X)\} \;\; p(A)q(A) = q(A)p(A).$$

H.P.(1) Prove (1).

As an illustration, here is a proof of the basic fact that *every* $A \in L(X)$ *with* $0 < \dim X < \infty$ *and* $\mathbb{F} = \mathbb{C}$ *has eigenvalues.* Indeed, there is $x \in X\backslash 0$ and, for any such x, $[x, Ax, A^2, \dots, A^{\dim X}x]$ must fail to be 1-1, hence there is $a \ne 0$ so that $p(A)x := \sum_j a(j)A^j x = 0$, showing that $p(A)$ fails to be 1-1, even though $p \ne 0$. Let $d := \max\{j : a(j) \ne 0\}$. Then, wlog, $a(d) = 1$, i.e., p is *monic*. Further, $d > 0$ since $x \ne 0$. Since $\mathbb{F} = \mathbb{C}$, we can therefore write p as the product $\prod_j (\cdot - z_j)$ of $d > 0$ linear factors. But, since $p(A) = \prod_j (A - z_j)$ fails to be 1-1, at least one of the factors $A - z_j$ must fail to be 1-1.

**** A-invariant direct sum decompositions ****

As a start toward a simplest matrix representation, assume that P is a **spectral projector** for the lm A, i.e., a lprojector that commutes with A,

$$PA = AP.$$

Then the corresponding direct sum decomposition

$$X = X_1 \dot{+} X_2, \qquad X_1 := \operatorname{ran} P,\; X_2 := \ker P$$

is **A-invariant** in the sense that its summands are A-invariant,

$$AX_i \subset X_i, \quad \text{all } i.$$

Conversely, for any such A-invariant direct sum decomposition $X = X_1 \dot{+} X_2$, the corresponding lprojector, given by $\operatorname{ran} P = X_1$, $\ker P = X_2$, is spectral for A since $\operatorname{ran}(AP) = AX_1 \subseteq X_1 = \operatorname{ran} P$, hence $PAP = AP$, while also $\operatorname{ran}(A(1-P)) = AX_2 \subseteq X_2 = \ker P$, hence $PA(1-P) = 0$, therefore, altogether, $PA = PAP + PA(1-P) = AP$.

If now X is finite-dimensional, then so are the X_i, and, with V_i any basis for X_i, $V := [V_1, V_2]$ is a basis for X with the happy property that $\hat{A} = V^{-1}AV = [V_1, V_2]^{-1}[AV_1, AV_2]$ is block-diagonal, in particular,

$$\hat{A} = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{bmatrix}, \qquad \hat{A}_i := V_i^{-1} A|_{X_i} V_i,$$

since the columns of AV_i are in X_i, hence their coordinates wrto V have nonzero entries only corresponding to the columns of V_i in V.

If you conclude from this that a search for 'simple' matrix representations for $A \in L(X)$ is equivalent to a search for A-invariant direct sum decompositions for A with many summands or, equivalently, a search for a many-termed sequence (P_i) in $L(X)$ with $P_iP_j = \delta_{ij}$ and $AP_i = P_iA$, all i, j, then you would be quite right.

**** primary decomposition ****

Here, for the record, is a first step in that direction that goes back to Frobenius. To be sure, this first step does not, in general, do the complete job. For that, just skip to the heading 'A finest A-invariant direct sum decomposition'.

Assuming $\dim X$ to be finite-dimensional, so is $L(X)$, hence $[A^r : r = 0, \ldots, \dim L(X)]$ cannot be 1-1, therefore there are polynomials $p \neq 0$ that **annihilate** A in the sense that $p(A) = 0$. Let p be any such *monic* annihilating polynomial and assume, for simplicity, that $\mathbb{F} = \mathbb{C}$. Then

$$p =: \prod_i p_i,$$

with $p_i = (\cdot - z_i)^{m_i}$, and $z_i \neq z_j$ for $i \neq j$. It follows that the polynomials

$$\ell_i := p/p_i, \quad \text{all } i,$$

do not have a common zero, hence

$$1 = \sum_i \ell_i h_i$$

for certain polynomials h_i. Indeed, let h_i be the unique polynomial of degree $< m_i$ for which $\ell_i h_i$ agrees m_i-fold with 1 at z_i. Then $1 - \sum_i \ell_i h_i$ is a polynomial of degree $< \sum_i m_i$ and vanishes m_i-fold at z_i, all i, hence must be zero.

Set

$$P_i := \ell_i(A) h_i(A), \quad X_i := \ker p_i(A), \qquad \text{all } i.$$

Then

$$1 = \sum_i P_i,$$

and, for $j \neq i$, P_j vanishes on X_i (since then ℓ_j has p_i as a factor), hence $1 = P_i$ on X_i, while $\operatorname{ran} P_i \subseteq X_i$ (since $p_i \ell_i h_i$ has p as a factor). Consequently, each P_i is a lprojector, with $\operatorname{ran} P_i = X_i$, and $P_j P_i = 0$ for $j \neq i$, giving us the direct sum decomposition

$$X = \oplus_i X_i,$$

which is A**-invariant** in the sense that $AX_i \subset X_i$ and, correspondingly, $AP_i = P_i A$, all i.

Of course, some of these summands may be trivial, unless we choose p a bit more carefully. In particular, choose p to be p_A, i.e., the **minimal (annihilating) polynomial** for A, i.e., an annihilating monic polynomial of smallest possible degree. Then $\ker(A - z_i)$ is not trivial (since, otherwise, $A - z_i$ would be invertible, hence already ℓ_i would be annihilating, contradicting the minimality of p_A). More than that,

$$X_i = \ker p_i(A) = \bigcup_r \ker(A - z_i)^r = \ker(A - z_i)^{m_i}.$$

Indeed, since $A - z_i$ is nilpotent on X_i, any $A - z_j = (z_i - z_j) - (z_i - A)) =: (z_i - z_j)(1 - N)$ for $j \neq i$ is 1-1 on X_i (since, for any nilpotent N, the sum $\sum_r N^r$ is well-defined since it is finite, while $\sum_r N_r(1 - N) = 1$, hence $1 - N$ is 1-1). Since also $A - z_i$ maps each X_j into itself, and the X_j are in direct sum, $(A - z_i)^r x = 0$ can hold only if $P_j x = 0$ for all $j \neq i$, i.e., for $x \in X_i$.

It follows that $\deg p_A \leq \dim X$ and that m_i is the degree of nilpotency of $A - z_i$ on X_i, i.e., m_i is the smallest integer r for which $\ker(A - z_i)^r = \ker(A - z_i)^{r+1}$.

It also follows that $A = z_i + (A - z_i)$ on X_i, with $(A - z_i)^{m_i} = 0$ there, hence

$$\exp(tA) = \exp(tz_i)\exp(t(A - z_i)) = \exp(tz_i) \sum_{r<m_i} t^r (A - z_i)^r / r! \quad \text{on } X_i,$$

thus providing a very helpful quite detailed description of the solution $y : t \mapsto \exp(tA)y_0$ of the aforementioned first-order ODE.

**** A finest A-invariant direct sum decomposition ****

Consider now an A-invariant direct-sum decomposition

$$X = X_1 \dot{+} \cdots \dot{+} X_r$$

that is **finest** in the sense that none of its summands can be split further into a *nontrivial* A-invariant direct-sum decomposition. This latter property can be shown (see H.P.(2)) to imply (assuming that $\mathbb{F} = \mathbb{C}$) for each summand X_i the existence of a scalar z_i and a vector $x_i \in X_i$ so that X_i has $V_i := [(A - z_i)^{q_i - j} x_i : j = 1, \ldots, q_i := \dim X_i]$ as a basis. The matrix representation with respect to the resulting basis $V := [V_i : i = 1, \ldots, r]$ for X is a **Jordan canonical form** for A.

H.P.(2) Let A be a lm on the n-dimensional ls X over $\mathbb{F} = \mathbb{C}$ and assume that any A-invariant direct sum decomposition of X has only the summands X and $\{0\}$. Prove the following:

(a) *For some* $z \in \mathbb{C}$, *there is* $x \in X$ *and* $q \in \mathbb{N}$ *so that, for the map* $N := A - z$, $N^{q-1}x \neq 0 = N^q x$.

(b) *With* N, x, *and* q *as in (a), there exists* $\lambda \in X'$ *so that* $\lambda N^{q-1} x \neq 0$, *and then* X *is the direct sum of* $\operatorname{ran} V$ *and* $\ker \Lambda^t$, *with* $V := [N^{q-j}x : j = 1, \ldots, q]$ *and* $\Lambda := [\lambda N^{i-1} : i = 1, \ldots, q]$. (Hint: $\Lambda^t V$ is 'obviously' invertible.)

(c) *Among the* q *satisfying (a), there is a largest one and, for that* q, $N^q = 0 \neq N^{q-1}$, *and* V *of (b) is a basis for* X, *hence* $q = \dim X$.

(d) *The matrix representation for* A *wrto* V *is a 'Jordan block', i.e., of the form* $z + [0, e_1, e_2, \ldots, e_{n-1}]$.

**** Jordan form ****

It is easy to verify that the Jordan form is unique up to permutation of its diagonal blocks (since the number and sizes of the blocks associated with a particular eigenvalue z_i are related in a 1-1 manner to the nondecreasing numerical sequence $(\dim\ker(A-z_i)^j : j=0,1,\ldots)$. In particular, not every lm A is diagonalizable, i.e., in this way **similar** to a diagonal matrix. However, the Jordan form has no practical importance since it does not depend continuously on A, hence cannot be constructed stably numerically.

For example, if $A = \begin{bmatrix} \pi & 1 \\ 0 & 3(\pi/3) \end{bmatrix}$, then, depending on the finite arithmetic used, the two diagonal entries of A may or may not be the same. If they are, then A is its own Jordan form, showing just one Jordan block. However, if they are not, then A is diagonable, and the Jordan form for A has two blocks.

**** Schur form ****

Instead (assuming without loss that X is an ips), one relies on the **Schur form** in which V is **unitary**, i.e., the basis is o.n., and $\hat{A}$ is 'only' upper triangular. Its construction still requires knowledge of the spectrum of A since $A - z = V(\hat{A} - z)V^{-1}$ implies that $\sigma(A) = \sigma(\hat{A})$ while the fact that $\hat{A}$ is upper triangular implies that $\sigma(\hat{A}) = \{\hat{A}(j,j) : j = 1,\ldots,n\}$. In order to discuss the standard derivation of this form, we need the hermitian of a lm.

**** c dual of a map ****

Having a matrix representation wrto an o.n. basis V has the advantage that the matrix reflects properties which involve the ip. For, having V o.n. is equivalent to having

$$\langle Va, Vb\rangle = \langle a, b\rangle \qquad \Big(= \sum_j a(j)\overline{b(j)}\ \Big). \tag{2}$$

In other words, V is **angle preserving**. Here and below, $\langle\cdot,\cdot\rangle$ is used to denote the relevant ip on whatever ips its arguments come from.

The interactions of $A \in bL(X,Y)$ with the inner products on X and Y are conveniently described in terms of the **c(onjugate) dual** or **Hermitian** A^c of A, which is the lm $A^c : Y \to X$ defined by

$$\langle Ax, y\rangle =: \langle x, A^c y\rangle, \qquad \text{all } x \in X, y \in Y. \tag{3}$$

In other words, $A^c y$ is the unique representer (with respect to the ip on X) for the bounded lfl $x \mapsto \langle Ax, y\rangle$, hence, by (VII.11)Riesz Representation Theorem, is well-defined for every $y \in Y$ in case X is complete. Note that A^c coincides with the dual $A^* : Y^* \to X^* : \lambda \mapsto \lambda A$ if we identify the Hs's X and Y with their duals X^* and Y^* via the isometric (but only skewlinear) map $v \mapsto \langle\cdot, v\rangle$. For,

$$A^* : \langle\cdot, y\rangle \mapsto \langle\cdot, y\rangle A = \langle A\cdot, y\rangle = \langle\cdot, A^c y\rangle.$$

In particular,

$$\|A^c\| = \|A\|.$$

H.P.(3) Verify that (3) defines a lm, and that A^c is the conjugate transpose of A, i.e., satisfies

$$A^c(r,s) = \overline{A(s,r)}, \quad \text{all } r, s,$$

in case $X = \mathbb{F}^n$ (with the scalar product as inner product).

**** Derivation of the Schur form ****

Let A be a lm on the n-dimensional ips X over the complex scalars. There is nothing to prove for $n = 1$. For $n > 1$, let (v_n, z) be an eigenpair for A^c (which exists since $\mathbb{F} = \mathbb{C}$), and extend v_n to an orthogonal basis $V = [v_1, \ldots, v_n]$ for X. Then $\langle Av_j, v_n\rangle = \langle v_j, A^c v_n\rangle = \overline{z}\langle v_j, v_n\rangle = 0$ for $j < n$, showing that $\{v_n\}\perp = \mathrm{ran}[v_1, \ldots, v_{n-1}] =: Y$ is A-invariant. Therefore, by induction, we may choose the orthogonal basis $W := [v_1, \ldots, v_{n-1}]$ for Y so that $W^{-1}A|_Y W$ is upper triangular. But then, with $V = [W, v_n]$, also $V^{-1}AV$ is upper triangular.

We will also need the following elegant twist: Assume, in addition, that also $C \in L(X)$ and that $AC = CA$. Then also $A^cC^c = C^cA^c$, hence (see H.P.(4)), we can choose v_n above to be an eigenvector for both A^c and C^c. The argument therefore supports the following.

(4) The Schur Form.Let A and C be lm's on the finite-dimensional ips X over $\mathbb{F} = \mathbb{C}$. *If A and C commute (i.e., $AC = CA$), then there exists an o.n. basis V for X so that both $V^{-1}AV$ and $V^{-1}CV$ are upper triangular.*

H.P.(4) Prove: *If X is a ls over $\mathbb{F} = \mathbb{C}$, and $A, C \in L(X)$ commute, then A and C have a common eigenvector.* (Hint: Show that, for any eigenvector y of C, the lss $\Pi(A)y := \mathrm{ran}[y, Ay, A^2y, \ldots]$ is A-invariant and any of its nonzero elements is an eigenvector for C.)

**** spectral theorem for normal matrices ****

Let $\hat{A} := V^{-1}AV$ for $A \in L(X)$, with V o.n. Then, $\langle \hat{A}a, b\rangle = \langle V^{-1}AVa, b\rangle = \langle AVa, Vb\rangle = \langle Va, A^cVb\rangle = \langle a, V^{-1}A^cVb\rangle$, hence

$$(\hat{A})^c = \widehat{(A^c)}.$$

Thus if A is **hermitian**, i.e., $A = A^c$, then also $\hat{A}^c = \hat{A}$. Hence, if, by (4)Schur, $\hat{A}$ is upper triangular, then $\hat{A}$ must be diagonal and its diagonal entries must be real. This proves the

(5) Spectral Theorem for hermitian matrices. *A hermitian map on a finite-dimensional ips is unitarily similar to a real diagonal matrix.*

Let now, more generally, A be **normal**, i.e., A and A^c commute. Then, by (4)Schur, we can so choose the o.n. basis V that both $\hat{A}$ and $\widehat{(A^c)} = \hat{A}^c$ are upper triangular. But that means that they must both be diagonal!

(6) Spectral Theorem for normal matrices. *A lm on a finite-dimensional ips is normal iff it is unitarily similar to a diagonal matrix.*

Eigenstructure of bounded lm's

If X is infinite-dimensional, then even the Schur form is, in general, not available. But, as we shall see, the spectral theorem still holds for normal lm's if they are also *compact*.

**** spectrum ****

Let now X be nls and $A \in bL(X)$. Then

$$\sigma(A) := \{z \in \mathbb{F} : A - z \text{ is not boundedly invertible}\}$$

is called the **spectrum** of A. It contains the point spectrum but need not coincide with it if X is infinite dimensional, since, in that case, $A - z$ may fail to be boundedly invertible even though it is 1-1.

(7) Example The simplest linear maps on $\mathbb{F}^n$ are the diagonal matrices. For more general function spaces $X \subseteq \mathbb{F}^T$, these correspond to **multipliers**, i.e., to maps of the form

$$M_a : f \mapsto af,$$

with $a \in \mathbb{F}^T$. Consider, in particular, $a \in X := C(T)$ with T compact metric, hence $M_a \in bL(X)$. I claim that

$$\sigma(M_a) = \operatorname{ran} a.$$

Indeed, since T is compact, so is $\operatorname{ran} a$, hence $\operatorname{ran} a$ is closed. Therefore, if $z \in \mathbb{F} \backslash \operatorname{ran} a$, then $1/(a - z) \in X$, hence $M_{1/(a-z)}$ is the bounded inverse for $M_{a-z} = M_a - z$, therefore $z \notin \sigma(M_a)$. Conversely, if $z \in \operatorname{ran} a$, then $z = a(t)$ for some t. But now, for any $\varepsilon > 0$, the function $f_\varepsilon : s \mapsto (1 - d(s,t)/\varepsilon)_+$ is in $X \backslash 0$, has norm 1, but has support only on $I := B^-_\varepsilon(t)$, hence $\|M_{a-z} f_\varepsilon\| \le \|(a - a(t))|_I\|_\infty = \omega_{a,t}(\varepsilon)$, therefore

$$\|M_{a-z} f_\varepsilon\| / \|f_\varepsilon\| = \omega_{a,t}(\varepsilon) \xrightarrow[\varepsilon \to 0+]{} 0$$

since a is continuous, hence M_{a-z} fails to be bounded below, and so $z \in \sigma(M_a)$.

But, while $\sigma(M_a) = \operatorname{ran} a$, M_a may fail to have eigenvalues. For, if $(M_a - z)f = 0$, then $f = 0$ on $T \backslash a^{-1}\{z\}$, hence z can be an eigenvalue for M_a only if $a^{-1}\{z\}$ is substantial enough to support a nontrivial continuous function. □

(8) Example Another instructive example is provided by the Volterra operator on $X := C[0\,..\,1]$, i.e., by the linear map given by

$$(Vf)(t) := \int_0^t f(s)\ \mathrm{d}s.$$

For any $z \in \mathbb{C} \backslash 0$, $V - z = (-z)(1 - V/z)$ is a second-kind integral operator, hence, by (VIII.10)Fredholm-Alternative, boundedly invertible if and only if it is 1-1. On the other hand, if $(V - z)f = 0$, then $zDf = f$ and $f(0) = 0$, i.e., f is a solution of a homogeneous ODE with homogeneous initial conditions, therefore $f = 0$. This shows that $V - z$ is 1-1 for every $z \in \mathbb{C}$. Thus, the only point possibly in the spectrum of V is 0, and it is indeed in the spectrum since V is compact, hence, although 1-1, cannot have a bounded inverse (or is directly seen not to be bounded below) (see H.P. (VIII.3)). □

The spectrum is closed since its complement is open, by (III.17)Proposition (since thereby the bounded invertibility of $A - z$ implies the bounded invertibility of $A - z'$ for every $z' \in B_{\|(A-z)^{-1}\|^{-1}}(z)$). By the same proposition,

$$\sigma(A) \subset B^-_{\|A\|},$$

since $|z| > \|A\|$ implies $\|z^{-1}\|^{-1} = |z| > \|A\| = \|z - (z - A)\|$.

**** resolvent ****

As $z \in \mathbb{C}\backslash\sigma(A)$ approaches $z' \in \sigma(A)$, $\|(A-z)^{-1}\|$ must go to infinity since, otherwise, $r := \limsup_{z\to z'} \|(A-z)^{-1}\| < \infty$, hence $A - z'$ would be boundedly invertible by (III.17)Proposition since there would be z (namely all z close enough to z') for which $\|(A-z') - (A-z)\| < \|(A-z)^{-1}\|^{-1}$. This implies that the spectrum $\sigma(A)$ is the set of singularities of the map

$$R : \mathbb{C}\backslash\sigma(A) \to bL(X) : z \mapsto (A-z)^{-1},$$

called the **resolvent** of A. It is continuous on its domain (since $\mathbb{C} \to bL(X) : z \mapsto A - z$ is). Therefore, R is bounded on any compact subset of $\backslash\sigma(A)$.

R is also differentiable, hence analytic, since

$$R(z) - R(z') = R(z)((A-z') - (A-z))R(z') = (z-z')R(z)R(z'),$$

therefore $\mathrm{d}R(z)/\,\mathrm{d}z = (R(z))^2$. This makes complex variable theory available as a convenient and powerful tool for the analysis of the resolvent and, ultimately, the spectral properties of A.

Remark If you are uncomfortable with operator-valued functions, consider instead the complex-valued function $z \mapsto \lambda R(z)x$ for arbitrary $\lambda \in X^*$ and arbitrary $x \in X$.

The spectrum $\sigma(A)$ is the set of singularities of R. In particular, R is analytic at infinity. Precisely, $A - z = -z(1 - A/z)$, hence the Neumann series

$$(A-z)^{-1} = -(1/z)\sum_{n=0}^{\infty}(A/z)^n \quad = -(1/z)\sum_{n=0}^{\infty}(A/r)^n\,(r/z)^n$$

converges for $|z| > r$ if also $\sup_n \|(A/r)^n\| < \infty$, hence converges for

$$|z| > \inf\{r : \sup_n \|(A/r)^n\| < \infty\} = \limsup \|A^n\|^{1/n} =: \varrho(A)$$

and so provides the Taylor series for R at infinity. (The equality in the last display is familiar from standard considerations concerning the radius of convergence of a power series: Let $c(r) := \sup_n \|(A/r)^n\|$. If $c(r) < \infty$, then $\|A^n\|^{1/n} \le rc(r)^{1/n}$, hence $\limsup_n \|A^n\|^{1/n} \le r \limsup_n c(r)^{1/n} = r$, showing the inf to be $\ge$ the lim sup. Conversely, if r equals the lim sup, then, for all $\varepsilon > 0$ and all $n \ge n_\varepsilon$, $\|A^n\|^{1/n} < r + \varepsilon$, i.e., $\|(A/(r+\varepsilon))^n\|^{1/n} < 1$, and therefore $\sup_n \|(A/(r+\varepsilon))^n\|^{1/n} < \infty$, showing the inf to be $\le$ the lim sup.)

In particular, since $\sigma(A)$ comprises the singularities of R, $\varrho(A)$ must equal the **spectral radius** of A, i.e.,

$$\varrho(A) = \sup|\sigma(A)|.$$

**** spectral radius ****

(9) Lemma. *X nls, $A \in bL(X)$. Then $\lim_n \|A^n\|^{1/n}$ exists, hence equals $\varrho(A)$, and $\forall\{k \in \mathbb{N}\}$ $\varrho(A) \le \|A^k\|^{1/k}$.*

Proof: For given $k \in \mathbb{N}$, write each $n \in \mathbb{Z}_+$ as $n = r(n) + m(n)k$, with $r(n), m(n) \in \mathbb{Z}_+$ and $r(n) < k$. Then $1 = r(n)/n + (m(n)/n)k$, therefore (since $r(n)/n \to 0$) we must have $m(n)/n \to 1/k$. This implies that

$$\|A^n\|^{1/n} = \|A^{r(n)+km(n)}\|^{1/n} \le \|A\|^{r(n)/n}\|A^k\|^{m(n)/n} \xrightarrow[n\to\infty]{} \|A^k\|^{1/k},$$

hence

$$\limsup \|A^n\|^{1/n} \le \|A^k\|^{1/k},$$

therefore

$$\limsup \|A^n\|^{1/n} \le \liminf \|A^k\|^{1/k}. \qquad \square$$

H.P.(5) Prove that $\varrho(A)$ does not change when we change the norm in X to an equivalent one.

H.P.(6) Prove directly, i.e., without recourse to complex function theory, that $\varrho(A) = \max|\sigma(A)|$ if A is a square matrix. (Hint: Use the Schur form and an appropriate diagonal matrix to show that among matrices similar to a given one are ones whose off-diagonal entries are as small as one pleases to make them.)

(10) Corollary. *If X is an ips and $A \in bL(X)$ is normal, then $\varrho(A) = \|A\|$.*

Proof: A normal $\implies \forall\{y \in X\}$ $\|Ay\|^2 = \langle Ay, Ay\rangle = \langle A^cAy, y\rangle = \langle AA^cy, y\rangle = \langle A^cy, A^cy\rangle = \|A^cy\|^2$, hence $\forall\{x \in X\}$ $\|A(Ax)\| = \|A^c(Ax)\|$, and therefore $\|A^2\| = \|A^cA\|$. On the other hand, $\forall\{C \in bL(X)\}$ $\|C\|^2 = \|C^cC\|$ since

$$\|C\|^2 = \sup_x |\langle C^cCx, x\rangle|/\|x\|^2 \le \sup_x \|C^cCx\|/\|x\| = \|C^cC\| \le \|C^c\|\|C\| = \|C\|^2.$$

Therefore $\|A^2\| = \|A\|^2$, hence $\varrho(A) = \lim \|A^n\|^{1/n} = \lim \|A^{2^n}\|^{2^{-n}} = \lim \|A\| = \|A\|$. $\square$

Note the fact just proved that *a blm A on an ips X is normal iff* $\forall\{x \in X\}$ $\|Ax\| = \|A^cx\|$.

**** isolating parts of the spectrum ****

(11) Theorem. *Let Γ be a simple closed curve in the complex plane that does not intersect $\sigma(A)$, hence partitions $\sigma(A)$ into $\sigma_1 \dot\cup \sigma_2$, with σ_1 the part inside Γ. Then*

$$P := \frac{-1}{2\pi\mathrm{i}} \int_\Gamma R(z)\,\mathrm{d}z \qquad (12)$$

is a bounded spectral projector for A, with $\sigma(A|_{\operatorname{ran} P})$ the part of $\sigma(A)$ in the interior of Γ, hence $\sigma(A|_{\ker P})$ the part of $\sigma(A)$ in the exterior of Γ.

In particular, (i) P a bounded lprojector that commutes with A, hence (ii) both $X_1 := \operatorname{ran} P$ and $X_2 := \operatorname{ran}(1-P) = \ker P$ are A-invariant closed lss's, and $X = X_1 \dot+ X_2$, and (iii) $A_j := A|_{X_j} \in bL(X_j)$, and (iv) $\sigma(A_j) = \sigma_j$.

Proof: (i) P is bounded (e.g., by $(|\Gamma|/2\pi)\max\|R(\Gamma)\|$). To see that P is a projector, note that P is unchanged if we deform Γ as long as we don't cross $\sigma(A)$ in the process. Therefore

$$P^2 = \frac{-1}{2\pi\mathrm{i}}\int_\Gamma \frac{-1}{2\pi\mathrm{i}}\int_{\Gamma'} R(z)R(z')\,\mathrm{d}z'\,\mathrm{d}z = \frac{-1}{2\pi\mathrm{i}}\int_\Gamma \frac{-1}{2\pi\mathrm{i}}\int_{\Gamma'} \frac{R(z)-R(z')}{z-z'}\,\mathrm{d}z'\,\mathrm{d}z$$
$$= \frac{-1}{2\pi\mathrm{i}}\int_\Gamma R(z)\,\mathrm{d}z = P,$$

the crucial equality by Cauchy's formula if we (as we may) choose Γ' close to Γ but enclosing it, hence

$$\int_{\Gamma'} \frac{R(z)}{z-z'}\,\mathrm{d}z' = R(z)\int_{\Gamma'} \frac{\mathrm{d}z'}{z-z'} = -2\pi\mathrm{i}R(z),$$

while

$$\int_{\Gamma} \frac{R(z')}{z-z'}\,\mathrm{d}z = R(z')\int_{\Gamma} \frac{\mathrm{d}z}{z-z'} = 0.$$

(ii) Since $R(z) = (A-z)^{-1}$ commutes with A, so does P, hence X_j is invariant under A (i.e., $AX_j \subset X_j$). In effect, $A = \operatorname{diag}\lceil A_1, A_2\rfloor$ with $A_j := A|_{X_j} \in bL(X_j)$ and $X = X_1 \dot{+} X_2$.

(iii) Therefore

$$R_j(z) := (A_j - z)^{-1} = R(z)|_{X_j}.$$

(iv) In particular, $\sigma(A_j) \subset \sigma(A)$, while if both $A_1 - z$ and $A_2 - z$ are boundedly invertible, then so is $A - z$. Therefore

$$\sigma(A_1) \cup \sigma(A_2) = \sigma(A).$$

On the other hand, for any z' outside Γ,

$$R(z')P = \frac{-1}{2\pi\mathrm{i}}\int_\Gamma R(z')R(z)\,\mathrm{d}z = \frac{-1}{2\pi\mathrm{i}}\int_\Gamma \frac{R(z')-R(z)}{z'-z}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{R(z)}{z'-z}\,\mathrm{d}z$$

which shows that $R_1 = R|_{X_1}$ has no singularities outside Γ. Correspondingly, R_2 has no singularities inside Γ. Hence, $\sigma(A_j) = \sigma_j$. □

Recall that we call such P a spectral projector for A, and note that it commutes with any spectral projector Q of A. Further, P and Q are **disjoint** (i.e., $PQ = QP = 0$) in case the curves used in their definition exclude one another.

Note that $P = 1$ in case $\sigma_2 = \{\}$. This is a special case of the assertion that

$$f(A) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(z)(z-A)^{-1}\,\mathrm{d}z \tag{13}$$

in case f is analytic on an open set containing $\sigma(A)$, and Γ is chosen in that open set and surrounding $\sigma(A)$. Note the exact analogue to Cauchy's formula.

Strictly speaking, (13) is an assertion only in case f is a polynomial or, more generally, a rational function, for then we have an alternative notion of what $f(A)$ might be. For more general f, (13) serves as a natural definition which is sensible since we can approximate such f uniformly by polynomials, hence obtain this definition as the limit.

**** isolated eigenvalues ****

The practically interesting case occurs when P is of finite rank. In that case, A_1 is a matrix, hence $\sigma_1 = \sigma(A_1)$ consists of finitely many points, each an eigenvalue of A, and we can compute the complete eigenstructure of A_1, including algebraic and geometric multiplicities of the eigenvalues.

In particular, each $z \in \sigma(A_1)$ is an *isolated* point of $\sigma(A)$, hence we can define

$$P_z := \frac{-1}{2\pi \mathrm{i}} \int_{\Gamma_z} (A - z')^{-1} \,\mathrm{d}z'$$

with Γ_z a simple closed curve enclosing z and excluding the rest of $\sigma(A)$. Its range $X_z := \operatorname{ran} P_z$ is finite-dimensional and equals $\cup_n \ker(A-z)^n$, i.e., consists of the generalized eigenvectors of A belonging to z. For this reason, X_z is also called the **generalized eigenspace of A belonging to z.**

The restriction $A_z := A|_{X_z}$ of A to X_z has the simple form $z+J_z$, with J_z a **nilpotent** map (i.e., $(J_z)^q = 0$ for some integer q).

H.P.(7) Why was the letter J used in the preceding sentence?

** numerics **

Further, if $C \in bL(X)$ is sufficiently close to A, then

$$P_C := \frac{-1}{2\pi \mathrm{i}} \int_{\Gamma_z} (C - z')^{-1} \,\mathrm{d}z'$$

will be close to P_z, hence so will $\operatorname{ran} P_C$ be to X_z. This shows that generalized eigenvectors of C belonging to eigenvalues of C inside Γ_z are close to those for A belonging to z and that even the algebraic multiplicities of these eigenvalues of C sum up to the algebraic multiplicity of z. These observations justify the approximate calculation of eigenpairs of A by the simple device of computing those of a nearby C and provide precise error estimates in terms of $\|A - C\|$.

Note that the required smallness of $\|A - C\|$ depends very much on how far away from $\sigma(A)$ Γ_z can be chosen. For, we require that, for all $z' \in \Gamma_z$ and all $t \in [0\,..\,1]$, the linear map $A - z' - t(C - A)$ be boundedly invertible. Only with a condition of this kind can we be assured that the spectrum of C captured by the contour Γ_z is comparable in its multiplicity and generalized eigenspace to that of the eigenvalue z of A. Thus we need, in effect, a condition like

$$\|A - C\| < \inf_{z' \in \Gamma_z} 1/\|(A - z')^{-1}\|.$$

On the other hand, to the extent that the piece of $\mathbb{C}$ enclosed by Γ_z is 'large', our computed information about the part of $\sigma(C)$ inside the contour may not be a very accurate description of z.

This approach becomes feasible only if C is particularly simple, or else if it has finite rank.

An *example* of the first kind might be the approximate calculation of the eigenstructure of a linear ordinary differential operator $A := D^m + \sum_{j<m} a_j D^j$ by the exact construction of part of the eigenstructure of the OD operator C obtained from A by replacing the coefficients by piecewise constant or piecewise linear approximants which makes it possible to construct elements in $\ker(C - z)$ exactly.

The approximation of A by a finite-rank C is the more customary procedure. Since it also requires that C approximate A uniformly, it is restricted to *compact* A.

**** eigenstructure of a compact lm ****

Let X Bs and $K \in bL(X)$ be compact. By (VIII.10)Fredholm-Alternative, any $z \in \sigma(K)\backslash 0$ is necessarily an eigenvalue of K, i.e.,

$$\sigma_P(K) \supset \sigma(K)\backslash 0.$$

Further, for each $z \in \sigma(K)\backslash 0$, $\dim\ker(K - z) < \infty$ by (VIII.8)Prop., i.e., each nonzero eigenvalue has finite multiplicity, and $\operatorname{ran}(K - z)$ is closed, by (VIII.9)Prop.

This implies

(14) Lemma. *$\sigma(K)\backslash 0$ consists of isolated points.*

Proof: The assumption that some $z \in \sigma(K)\backslash 0$ is not isolated implies the existence of a sequence (z_n) of distinct points in $\sigma(K)$ with $c := \inf_n |z_n| > 0$. By H.P.(VIII.4), this contradicts the compactness of K. □

It follows that each $z \in \sigma(K)\backslash 0$ has an associated finite-dimensional generalized eigenspace X_z and that the corresponding spectral projector

$$P_z := \frac{-1}{2\pi \mathrm{i}} \int_{\Gamma_z} (K - z')^{-1}\,\mathrm{d}z'$$

has as its interpolation functionals the range of its dual, i.e., the range of

$$P_z^* = \frac{-1}{2\pi \mathrm{i}} \int_{\Gamma_z} (K^* - z')^{-1}\,\mathrm{d}z',$$

hence this consists of the generalized eigenvectors of K^* belonging to z.

Spectral Theorem for compact normal maps

Let X Hs and $A \in bL(X)$ be *normal.* Then each spectral projector

$$P = \frac{-1}{2\pi \mathrm{i}} \int_{\Gamma} (A - z)^{-1}\,\mathrm{d}z$$

commutes with its conjugate

$$P^c = \frac{-1}{2\pi \mathrm{i}} \int_{\overline{\Gamma}} (A^c - z)^{-1}\,\mathrm{d}z,$$

hence is an *orthogonal* projector. Note the appearance of $\overline{\Gamma}$ here; the corresponding reversal of direction accounts for the fact that there is again $(-1/2\pi\mathrm{i})$ in front of the integral, rather than its conjugate.

H.P.(8) Prove that a normal lprojector is an orthogonal projector.

This implies that, for a normal compact lm $K \in bL(X)$, every P_z for $z \in \sigma(K)\backslash 0$ is an orthogonal projector. Further, $\operatorname{ran} P_z = X_z = \ker(K - z)$, since $K_z = K|_{X_z}$ is normal and has the single eigenvalue z, hence $KP_z = zP_z$. Since

$$P_z P_{z'} = P_{z'} P_z = \delta_{z,z'},$$

any finite sum $Q := \sum_{z\in Z} P_z$ is also an orthogonal projector and $K = \sum_{z\in Z} zP_z$ on $\operatorname{ran} Q$, and $\sigma(K(1-Q)) = \sigma(K)\backslash Z$. Therefore, $\|K - KQ\| = \varrho(K - KQ) = \max|\sigma(K)\backslash Z|$.

If $\#\sigma(K) < \infty$, this proves that

$$K = \sum_{z\in\sigma(K)\backslash 0} zP_z.$$

In the contrary case, 0 is the only cluster point of $\sigma(K)$, hence is the limit of the sequence (z_j) that contains the eigenvalues of K ordered by magnitude. Now we have

$$\|K - \sum_{j<n} z_j P_{z_j}\| = |z_n| \xrightarrow[n\to\infty]{} 0,$$

hence

$$K = \sum_j z_j P_{z_j}$$

in this (norm) sense.

An alternative formulation of this Spectral Theorem for compact normal maps makes explicit use of an o.n. basis, i.e., writes P_z explicitly as $V_z V_z^c = \sum_{v\in V_z} [v]\langle\cdot, v\rangle$, with V_z any o.n. basis for $X_z = \ker(K - z)$. This gives the formula

$$K = \sum_n \sum_{v\in V_{z_n}} z_n [v]\langle\cdot, v\rangle.$$

In order to obtain the exact analogue of the spectral theorem for normal matrices, let (v_j) be the sequence made up of the $v \in V_{z_n}, n = 1, 2, \ldots$, in such a way that

$$v_j \in V_{z_n}, \quad v_k \in V_{z_m}, \quad j < k \quad \Longrightarrow \quad n \le m.$$

Then

$$K = V\hat{K}V^c,$$

with V the unitary map

$$V : \ell_2 \to X : a \mapsto \sum_j v_j a(j)$$

and

$$\hat{K} = \operatorname{diag}\lceil \ldots, \underbrace{z_n, \ldots, z_n,}_{\dim X_{z_n}\ \text{times}} , \ldots \rfloor.$$

The SVD for compact maps

The **Singular Value Decomposition**, or **SVD** for short, of a matrix has become a powerful tool for the analysis of 'incorrect' linear systems, i.e., systems which are either not square or else are singular or nearly singular in some sense. The same tool is available for a compact $K \in L(X, Y)$ with X, Y Hs's. For, both K^cK and KK^c are positive semidefinite, hence all their eigenvalues are nonnegative. Therefore, the spectral theorem for compact normal lm's permits us to write

$$K^cK = U\Sigma^2U^c, \qquad KK^c = V\Sigma^2V^c,$$

hence $KU\Sigma^2U^c = KK^cK = V\Sigma^2V^cK$, therefore (multiplying from the left by V^c and from the right by U)

$$V^cKU\Sigma^2 = \Sigma^2V^cKU,$$

with Σ a diagonal matrix with a nonnegative nonincreasing diagonal so that Σ^2 contains in its diagonal in nonincreasing order the elements of $\sigma(K^cK) = \sigma(KK^c)$, i.e., the squares of the **singular values** of K. This implies that V^cKU is block-diagonal,

$$V^cKU = \text{ diag}\lceil \ldots, E_j, \ldots \rfloor$$

say, with different blocks belonging to different singular values. The matrix version of the SVD theorem provides, for each such diagonal block E_j, unitary matrices U_j and V_j of the same size so that $V_j^cE_jU_j$ is diagonal. Hence, altogether

$$S \text{ diag}\lceil \ldots, V_j, \ldots \rfloor V^cKU \text{ diag}\lceil \ldots, U_j, \ldots \rfloor$$

is diagonal and nonnegative, the diagonal matrix S having been chosen appropriately. It then follows that this diagonal matrix must be Σ, hence

$$K = A\Sigma C^c,$$

with

$$A := V \text{ diag}\lceil \ldots, V_j, \ldots \rfloor S^c, \quad C := U \text{ diag}\lceil \ldots, U_j, \ldots \rfloor$$

both unitary.