A classification of transitive ovoids, spreads, and m-systems of polar spaces

John Bamberg and Tim Pentilla
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Abstract. Many of the known ovoids and spreads of finite polar spaces admit a transitive group of collineations, and in 1988, P. Kleidman classified the ovoids admitting a 2-transitive group. A. Gunawardena has recently extended this classification by determining the ovoids of the seven-dimensional hyperbolic quadric which admit a primitive group. In this paper we classify the ovoids and spreads of finite polar spaces which are stabilised by an insoluble transitive group of collineations, as a corollary of a more general classification of m-systems admitting such groups.


1 Introduction

The objects of classical polar spaces known as ovoids and spreads have played a central role in finite geometry in modern times. To use the words of J. A. Thas [Tha01], they have “many connections with and applications to projective planes, circle geometries, generalized polygons, strongly regular graphs, partial geometries, semi-partial geometries, codes, designs.” An ovoid of a finite polar space \( P \) is a subset of the point-set of \( P \) that has exactly one point in common with each generator of \( P \). Dually, a spread of \( P \) is a set of generators which form a partition of the points of \( P \). It is well known (see [Tha01]) that the ovoids and spreads of \( P \) have a common number of elements, which we denote by \( \mu(P) \).

An ovoid (resp. spread) of \( P \) is transitive if there exists a group of collineations of \( P \) which stabilises and acts transitively on the ovoid (resp. spread). The doubly transitive ovoids were classified by Peter Kleidman [Kle88a], and the primitive ovoids of the seven-dimensional hyperbolic quadric \( Q_4^+(q) \) have been classified by Athula Gunawardena [Gun00]. In 1987, Bagchi and Sastry [BS87] proved that the only transitive ovoids of \( W_4(q) \), for \( q \) even, are the elliptic quadric and the Suzuki-Tits ovoid. In 2003, Cossidente and Korchmáros [CK03] classified the linearly transitive ovoids of the Hermitian surface \( H_q(2) \) for \( q \) even. In this paper, we extend the aforementioned results to ovoids of finite polar spaces admitting an

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insoluble transitive group. This is a corollary of a classification of a much wider class of geometries, transitive m-systems.

In 1994, Shult and Thas [ST94] introduced a natural generalisation of an ovoid and spread which they called an m-system. In a finite polar space $\mathbb{P}$, any collection of m-subspaces of $\mathbb{P}$ which are pair-wise opposite has its cardinality bounded by $\mu(\mathbb{P})$, the number of elements of any ovoid or spread of $\mathbb{P}$. Thus an m-system is such a collection of m-subspaces with exactly $\mu(\mathbb{P})$ elements. A comprehensive treatment of the properties of m-systems can be found in the thesis of Deirdre Laycock [Luy02].

A concise description of the classification of transitive m-systems (with insoluble stabilisers) is given below, which are described via seeding objects and various construction mechanisms. These will be discussed in the forthcoming sections of this paper.

**Theorem 1.1.** Let $\mathcal{M}$ be an m-system of a finite classical polar space $\mathbb{P}$ of rank at least 2, which admits an insoluble transitive group of automorphisms. Then $\mathcal{M}$ is given in the following table:

<table>
<thead>
<tr>
<th>m-systems</th>
<th>Description</th>
<th>S</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical of $\mathbb{P}$</td>
<td>PSU$_3(q^2)$</td>
<td>Embedding of $\mathbb{P}_2(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $H_3(2^2)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(2^2)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $H_3(2^2)$</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>m-systems</th>
<th>Description</th>
<th>S</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptic quadric in $W_5(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of projective line $W_5(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Suzuki-Tits ovoid of $W_5(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Embedding of $Q_1^+(q)$, $q$ even</td>
<td></td>
</tr>
<tr>
<td>LÃ¼nke ovoid of $W_5(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(2^2)$</td>
<td></td>
</tr>
<tr>
<td>LÃ¼nke ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Dual of classical ovoid of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Herwig ovoid of $W_5(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Dual of exceptional ovoid of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $Q_4(q)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $Q_4(q)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $Q_4(q)$</td>
<td></td>
</tr>
<tr>
<td>Exceptional of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>m-systems</th>
<th>Description</th>
<th>S</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptic quadric in $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Embedding of $Q_5^+(q)$</td>
<td></td>
</tr>
<tr>
<td>Regular spread in $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>Suzuki-Tits ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>LÃ¼nke ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>Patterson ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>Ree-Tits ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>Thas-Kantor ovoid of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>m-systems</th>
<th>Description</th>
<th>S</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thas-Kantor spread of $Q_4(q^2)$</td>
<td>PSU$_3(q^2)$</td>
<td>$\mathcal{H}(q)$-planes on Thas-Kantor ovoid, $q = 3^k$</td>
<td></td>
</tr>
<tr>
<td>Ree-Tits 1-system of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$\mathcal{H}(q)$-planes on Ree-Tits ovoid, $q = 3^k$</td>
<td></td>
</tr>
<tr>
<td>Ree-Tits spread of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$\mathcal{H}(q)$-planes on Ree-Tits ovoid, $q = 3^k$</td>
<td></td>
</tr>
<tr>
<td>Exceptional 1-system of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Exceptional spread of $\mathcal{H}(3)$</td>
<td></td>
</tr>
<tr>
<td>Derived 1-system of $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Derivation of exceptional spread of $\mathcal{H}(3)$</td>
<td></td>
</tr>
<tr>
<td>m-system of $Q_5^+(q)$ embedded in $Q_4(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(q^2)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>m-systems</th>
<th>Description</th>
<th>S</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic Quads</td>
<td>Ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q &gt; 3$</td>
</tr>
<tr>
<td>Spread of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q &gt; 3$</td>
<td></td>
</tr>
<tr>
<td>Elliptic quadric in $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Embedding of $Q_5^+(q)$</td>
<td></td>
</tr>
<tr>
<td>Hall ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Image under the Klein map of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Unitary 1-system of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Dye ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Cooperstein ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>Field reduction of $H_3(q^2)$</td>
<td></td>
</tr>
<tr>
<td>Kantor's desarguesian ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q$ even</td>
<td></td>
</tr>
<tr>
<td>Kantor's unitary ovoid of $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td>$q \equiv 2 \mod 3$</td>
<td></td>
</tr>
<tr>
<td>m-system of $Q_6(q)$ embedded in $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m-system of $Q_6(q)$ embedded in $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>m-system of $Q_6(q)$ embedded in $Q_5^+(q)$</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>obtained by triality of one of the above</td>
<td>PSU$_3(q^2)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Gunawardena showed that the only ovoid of the hyperbolic quadric $Q_7(q)$ that is primitive and not 2-transitive is the Cooperstein ovoid. We have the following extension of Gunawardena and Kleidman's work. (Note: A subgroup $G$ of $PGL_d(q)$ is one-dimensional semilinear if it can be embedded in $GL_1(q^2)$.)

**Corollary 1.2.** Let $\mathcal{M}$ be an ovoid of a finite classical polar space $\mathbb{P}$, stabilised by a transitive group $G$ that is not one-dimensional semilinear and not 2-transitive. Then $\mathcal{M}$ is one of:

(i) the Exceptional Ovoid of $H_3(2^2)$ (which is imprimitive),
(ii) the Hall Ovoid of $Q_4^+(3)$ (which is imprimitive), or
(iii) the Cooperstein Ovoid of $Q_4^+(5)$ (which is primitive).

If $G$ in the above corollary were primitive and one-dimensional semilinear, then $G$ would be solvable, and hence be of affine type. Moreover, the socle of $G$ would have prime order, thus rendering the degree of $G$ to be prime. The size of an m-system is one more than a prime power, which implies that the degree is a Fermat prime. Therefore, for $q$ odd, the Cooperstein ovoid is the only ovoid admitting a primitive group $G$, where it is permitted that $G$ be one-dimensional semilinear.

2 Some notation and background theory

A prime number $r$ dividing $q^r - 1$ is a prime primitive divisor of $q^r - 1$ if $r$ does not divide $q^r - 1$ for $r$ smaller than $e$, or equivalently, $q$ has order $e$ modulo $r$. Zsigmondy proved in [Zsi92] that if $q$ is an integer greater than 1, and if $e$ is a positive integer such that $q^e - 1$ has
no primitive prime divisors, then \( q^f = p^f = 2^b \) or \( e = 2 \). A divisor \( r \) of \( q^f - 1 \) that is coprime to each \( q^i - 1 \) for \( i < e \) is said to be a primitive divisor, and we call the largest primitive divisor \( \Phi_k(q) \) of \( q^f - 1 \) the primitive part.

The geometric setting of this paper is in the finite classical polar spaces; those geometries which come from a finite vector space equipped with a reflexive sesquilinear form or quadratic form. We will assume that the reader is familiar with the fundamental theory of polar spaces and we will use projective notation for polar spaces so that they differ from the standard notation for their collineation groups. For example, we will use the notation \( W_{d-1}(q) \) to denote the symplectic polar space coming from the vector space \( V_d(q) \) equipped with a non-degenerate alternating form. Here is a summary of the notation we will use for polar spaces, together with their ovoid numbers.

### Table 2: Notation for the finite polar spaces, together with their ovoid numbers.

<table>
<thead>
<tr>
<th>Polar Space</th>
<th>Notation</th>
<th>Collineation Group</th>
<th>Ovoid Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symplectic</td>
<td>( W_{d-1}(q) ), d even</td>
<td>( P S_p(q) )</td>
<td>( q^{d/2} + 1 )</td>
</tr>
<tr>
<td>Hermitian</td>
<td>( H_{d-1}(q^2), d ) odd</td>
<td>( U_O(q^2) )</td>
<td>( q^2 + 1 )</td>
</tr>
<tr>
<td>Hermitian</td>
<td>( H_{d-1}(q^2), d ) even</td>
<td>( U_O(q) )</td>
<td>( q^{d-1/2} + 1 )</td>
</tr>
<tr>
<td>Orthogonal, elliptic</td>
<td>( Q_{d-1}(q^2), d ) odd</td>
<td>( O(q) )</td>
<td>( q^{d-1/2} + 1 )</td>
</tr>
<tr>
<td>Orthogonal, parabolic</td>
<td>( Q_{d-1}(q), d ) odd</td>
<td>( O(q) )</td>
<td>( q^{d-1/2} + 1 )</td>
</tr>
<tr>
<td>Orthogonal, hyperbolic</td>
<td>( Q_{d-1}(q), d ) even</td>
<td>( O(q) )</td>
<td>( q^{d-1/2} + 1 )</td>
</tr>
</tbody>
</table>

Throughout, two sets of subspaces \( S_1 \) and \( S_2 \) of a common polar space will be said to be equivalent if there exists a collineation of the polar space \( S_1 \) onto \( S_2 \). The group theoretic counterpart is the notion of quasi-equivalent representations. We will say that two representations \( \varphi_1 : G \rightarrow GL(V) \) and \( \varphi_2 : G \rightarrow GL(V) \) are quasi-equivalent if there exists an automorphism \( \tau \) of \( G \) such that \( \varphi_1 \) is equivalent to \( \tau \varphi_2 \). Hence, if \( \varphi_1 \) and \( \varphi_2 \) are quasi-equivalent, then \( \varphi_1(G) \) and \( \varphi_2(G) \) are conjugate in \( GL(V) \).

In the next four sections, we outline the known methods of obtaining new m-systems from old ones.

### 2.1 Field reduction

Let \( E \) be a field extension of \( F \) of degree \( b \), and let \( Tr_{E/F} : E \rightarrow F \) be the relative trace map. The \( d \)-dimensional vector space \( V_{d}(F) \) over \( F \) can be thought of as a \( d/b \)-dimensional \( E \)-vector space if one identifies the \( d/b \)-vectors of \( F^b \) with the \( d \)-vectors of \( F \). Hence, the linear transformations of \( V_{d/b}(E) \) induce linear transformations of \( V_d(F) \) (but not conversely). Thus, we can get the embedding \( GL_{d/b}(E) \leq GL_d(F) \). Now suppose that \( V_{d/b}(E) \) is equipped with a non-degenerate symplectic form \( \kappa \). Consider the composition \( Tr_{E/F} \circ \kappa \) and observe that for all \( v, w \in V_d(F) \), we have

\[
Tr_{E/F}(\kappa(v, w)) = Tr_{E/F}(-\kappa(w, v)) = -Tr_{E/F}(\kappa(w, v)).
\]

Hence \( Tr_{E/F} \circ \kappa \) is a non-degenerate symplectic form on \( V_d(F) \), and an isometry of \( (V_{d/b}(E), \kappa) \) induces an isometry of \( (V_d(F), Tr_{E/F} \circ \kappa) \). Therefore, we have the embedding \( Sp_{d/b}(E) \leq Sp_d(F) \). Field reduction of this kind is ubiquitous in polar geometry, particularly in the construction of m-systems. There are similar 'passages' between polar spaces which map m-systems to m'-systems, where \( m \) and \( m' \) may be different. This correspondence has the property that if \( S \) is a subspace of a polar space \( P_1 \) of dimension \( m \), then the image of that subspace is a subspace of a polar space \( P_2 \) of dimension \( m' \) determined by \( m \). A comprehensive study of field reduction can be found in [GIL] where the author investigates when a polar space over \( GF(q^f) \) defined by a reflexive sesquilinear form \( f \) (resp. quadratic form \( Q \)), together with a \( GF(q^f) \)-linear map \( L: GF(q^f) \rightarrow GF(q) \), gives rise to a non-degenerate polar space with reflexive sesquilinear form \( L \circ f \) (resp. quadratic form \( L \circ Q \)). In the table below, we give a brief account of when field reduction of an m-system yields an \( m' \)-system (for some \( m' \)).

### Table 3: Field reduction of m-systems.

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( m' )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{d-1}(q^2) )</td>
<td>( W_{d-1}(q) )</td>
<td>( b(n + 1) )</td>
<td>[HJ00, Theorem 3]</td>
</tr>
<tr>
<td>( H_{d-1}(q^2), d ) even</td>
<td>( W_{d-1}(q^2) )</td>
<td>( 2(n + 1) )</td>
<td>[ST94, Theorem 14]</td>
</tr>
<tr>
<td>( H_{d-1}(q^2), d ) odd</td>
<td>( H_{d-1}(q^2) )</td>
<td>( b(n + 1) )</td>
<td>[ST95, § 9]</td>
</tr>
<tr>
<td>( H_{d-1}(q^2), d ) even</td>
<td>( O(q) )</td>
<td>( 2(n + 1) )</td>
<td>[ST94, Theorem 12(a)]</td>
</tr>
<tr>
<td>( Q_{d-1}(q^2) )</td>
<td>( O(q) )</td>
<td>( b(n + 1) )</td>
<td>[HJ00, Theorem 2]</td>
</tr>
<tr>
<td>( Q_{d-1}(q), d ) even</td>
<td>( O(q) )</td>
<td>( 2(n + 1) )</td>
<td>[ST94, Theorem 12(b)]</td>
</tr>
<tr>
<td>( Q_{d-1}(q), d ) odd</td>
<td>( O(q) )</td>
<td>( 2(n + 1) )</td>
<td>[ST94, Theorem 13(a)]</td>
</tr>
<tr>
<td>( Q_{d-1}^+(q) ), d even</td>
<td>( Q_{d-1}^+(q) )</td>
<td>( 2(n + 1) )</td>
<td>[ST95, § 9]</td>
</tr>
</tbody>
</table>

### 2.2 Embeddings

Suppose \( P \) is a finite polar space, and \( \mathcal{H} \) is a non-degenerate hyperplane section of \( P \). Then two totally singular (isotropic) subspaces of \( \mathcal{H} \) are opposite if and only if they are opposite in the larger space \( P \). Therefore, if the ovoid numbers of both spaces are equal, then an m-system of \( \mathcal{H} \) embeds as an m-system of \( P \). This natural method of obtaining m-systems is summarised in the table below.

### Table 4: Natural embeddings of m-systems.

<table>
<thead>
<tr>
<th>( F )</th>
<th>( P_2 )</th>
<th>Common Ovoid Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{d-1}(q^2) )</td>
<td>( Q_{d-1}(q) )</td>
<td>( q^{d/2} + 1 )</td>
</tr>
<tr>
<td>( Q_{d-1}(q) )</td>
<td>( Q_{d-1}^+(q) )</td>
<td>( q^{d/2} + 1 )</td>
</tr>
<tr>
<td>( H_{d-1}(q^2) )</td>
<td>( H_{d-1}(q^2) )</td>
<td>( 2q^{d/2} + 1 )</td>
</tr>
</tbody>
</table>

1 The term 'passage' was coined by Shult and Thus [ST95, § 9] for a correspondence induced by the field reduction (with perhaps some normalisation by a scalar) of a non-degenerate polar space yielding another non-degenerate polar space.
2.3 Triality

The hyperbolic quadric $Q_7^i(q)$ is unusual in that it admits a graph automorphism of order 3 known as a triality (c.f., [Tha59]). We describe this correspondence in what follows. The totally singular solids of $Q_7^i(q)$, that are, the maximal totally singular subspaces of $Q_7^i(q)$, fall into two classes according to the equivalence relation that pairs distinct totally singular solids are disjoint or intersect in a line. Let $\Sigma_1$ and $\Sigma_2$ be these two classes of solids, and let $\mathcal{P}$ be the set of points of $Q_7^i(q)$. Then a triality is an incidence preserving map $\tau$ on the subspaces of $Q_7^i(q)$ which maps points to $\Sigma_1$, maps $\Sigma_1$ to $\Sigma_2$, maps $\Sigma_2$ to $\mathcal{P}$, and preserves the set of totally singular lines $\mathcal{L}$, with the extra condition that $\tau$ have order 3. Hence a triality induces a 3-cycle on the Dynkin diagram of $PG_7^i(q)$.

![Diagram of $PG_7^i(q)$](image)

Figure 1 The Dynkin Diagram of $PG_7^i(q)$.

Moreover, since triality preserves incidence, it is not difficult to show that two opposite subspaces of $Q_7^i(q)$ of the same dimension are mapped by a triality to another pair of opposite subspaces. Hence with a triality, we obtain a spread from an ovoid, a spread from a spread, an ovoid from a spread, and a 1-system from a 1-system. However, in the case that the two $m$-systems are of the same type, equivalence is not necessarily preserved. Therefore, new $m$-systems can arise by the application of a triality automorphism.

2.4 Derivation

It is well known that one can obtain new spreads of projective space from old spreads by replacing a regulus with its opposite regulus (see [Dem97]). Similarly, Payne and Thas [PT84] introduced a method commonly known as derivation whereby new ovoids of $H_3(q^2)$ arise from a classical ovoid (see Section 3.1). That is, given a classical ovoid $C$ of $H_3(q^2)$ and a line $\ell \in PG_3(q^2)$ such that $\ell \cap C$ is contained in $\ell$, we get a new ovoid by removing the points of $C$ on $\ell$ and replacing them with the totally singular points on $\ell^\perp$. Here we describe a generalisation of this technique which can be found in Luyckx's thesis [Luy02, Chapter 8].

Let $M$ be an $m$-system of a finite polar space $P \in PG_6(q)$ with elements $\{x_1, x_2, \ldots, x_t\}$. Suppose there is a projective subspace $\ell \in PG_6(q)$ of dimension $2m + 1$ containing the first $s$ elements of $M$, where $s < t$. If there exist disjoint totally singular $m$-subspaces $\{x'_1, x'_2, \ldots, x'_s\}$ each contained in $\ell$, such that $U_{\ell \cap x'_i}$ covers the same points as $U_{\ell \cap x_i}$, then the set

$$\{x'_1, x'_2, \ldots, x'_s, x_{s+1}, \ldots, x_t\}$$

forms an $m$-system of $P$.

3 Known examples of transitive $m$-systems

3.1 The Hermitian varieties

Classical ovoids of $H_3(q^2)$

There are two equivalent ways to view the classical ovoids of $H_3(q^2)$ (also known as Hermitian curves). One way to construct a classical ovoid is to take a plane that is not tangent to $H_3(q^2)$ from which the intersection is an ovoid. Also, one can construct a classical ovoid by considering the absolute points of a (non-degenerate) unitary polarity of $H_3(q^2)$. The classical ovoids of $H_3(q^2)$ are pairwise isometric. Now for all $q$, the group $PSU_3(q^2)$ acts 2-transitively on the classical ovoid of $H_3(q^2)$. In the case that $q = 5$, we have that $A_7$ (which is a maximal subgroup of $PSU_3(5^2)$) acts transitively on the classical ovoid.

Exceptional ovoid of $H_3(5^2)$

As it was for the classical example, the case $q = 5$ is rather special. A line of the ambient projective plane of the hermitian curve $H_3(q^2)$ meets this curve in 1 or $q + 1$ points. The latter are known as secant lines of $H_3(q^2)$. A set of $q^2 - q + 1$ secant lines which partition the points of $H_3(q^2)$ is a initial spread, and there exists an interesting example for $q = 5$. For $H_3(5^2)$, there exists a unital spread admitting $PSL_2(7)$, which is in turn, a subgroup of the maximal subgroup $A_7$ of $PSU_3(5^2)$ (see [CP05, §4]). By taking the perps of these lines inside $H_3(5^2)$, we obtain an ovoid spanning the entire hermitian surface for which the full stabiliser $G = Z_3 \times (Z_3 \times PSL_2(7))$ acts transitively but imprimitively on the ovoid. It is important to note that $G$ stabilises a non-degenerate hyperplane and that the simple group $PSL_2(7)$ is intransitive on the ovoid (it has three orbits of size 42). We will call this ovoid the exceptional ovoid of $H_3(5^2)$. We should also mention here that since $G$ stabilises a non-degenerate hyperplane of $H_3(5^2)$, the dual spread of $Q_7^i(5)$ admits a transitive group which does not appear as a nearly simple subgroup of $PGO_7^i(5)$. By inspecting Table 3 (see also [Gill]), one can readily see that field reduction of $H_3(q^2)$ can only produce an hermitian non-degenerate polar space (when $q$ is a square). Thus we do not obtain any other $m$-systems from this example for higher dimensional polar spaces.

Singer-type ovoid of $H_3(q^2)$

For $q$ even, there is a non-classical transitive ovoid of $H_3(q^2)$ admitting $Z_{q^2+1} : Z_3$, which we call the Singer type ovoid, in the tradition of [CK03]. It was first constructed by Baker, Ebert, Korchmáros, and Szonyi [BEKS93] by "multiple derivation" from a cyclic chiral spread of $H_3(q^2)$ and a classical ovoid $H_3(q^2)$ of $H_3(q^2)$. This is an example of a transitive $m$-system with a one-dimensional semi-linear stabiliser.

Uniqueness and non-existence results

It was proved by Thas [Tha81] that there are no spreads of $H_3(q^2)$ and there are no ovoids of $H_{2n}(q^2)$ for $n > 1$. The same author in [Tha92] proved that there are no spreads of $H_{2n+1}(q^2)$ for $n > 1$. The existence of spreads of $H_4(q^2)$ is an open question. This conjecture has been

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2 We thank A. Cossidente for pointing this example out.
settled by Brouwer for \( q = 2 \) in an unpublished computer result. Cossidente and Korchmáros [CK03] have classified the \textit{linearly} transitive ovoids of the Hermitian surface \( H_3(q^2) \) for \( q \) even. By a "linearly transitive ovoid", they mean an ovoid for which there exists a linear group (as opposed to a semilinear group of collineations) acting transitively on the ovoid. They showed that such an ovoid is classical or the Singer-type ovoid. It was already shown, by Brouwer and Wilbrink [BW90], that all transitive ovoids of \( H_3(2^2) \) are classical (they classified all ovoids of \( H_3(2^2) \)). Hamilton and Mathon (see [HM01, Corollary 1]) proved that there are no \( m \)-systems of \( H_3(q^2) \) for \( n > 2m + 1 \). So in particular, this proves Thas' result that there are no ovoids of \( H_3(q^2) \) for \( n > 1 \). Recently, Jan De Beule and Klaus Metsch [DBM05] have proved that \( H_3(2^3) \) has no ovoids. For \( H_3(2^3) \), we will need the following observation:

**Lemma 3.1.** Up to isometry, there is just one ovoid of \( H_3(2^3) \) admitting \( \text{PSL}_2(7) \) and it is classical.

**Proof.** It is well known (see [JLPW95]) that there is up to quasi-equivalence a unique unitary irreducible representation of \( \text{PSL}_2(7) \) in characteristic 3. Moreover, by computer we see that it has the following orbit lengths on points of \( H_3(3^2) \): 28, 42, 42, 168. However, the smallest orbit here, which has the size of an ovoid, is not an ovoid (in fact the lines of \( H_3(3^2) \) meet this point-set in 0, 1, or 4 points).

The only other equivalence class of representations of \( \text{PSL}_2(7) \) in \( \text{PGU}_3(3^2) \) are those which fix a non-degenerate hyperplane of \( H_3(3^2) \). Let \( O \) be a classical ovoid of \( H_3(3^2) \) admitting \( \text{PSL}_2(7) \). First note that \( \text{PSL}_2(7) \) acts transitively on \( O \). By computer inspection, an element of \( \text{PSL}_2(7) \) of order 3 has exactly one fixed point in \( H_3(3^2) \). So a fixed point of such an element of order 3 lies in \( O \) as \( |O| = 28 \). Therefore, an element of order 3 acts fixed point freely on the other orbits of \( \text{PSL}_2(7) \), and so these other orbits have size divisible by 3. So there is at least one orbit of \( \text{PSL}_2(7) \) with length coprime to 3, that is, \( O \) is the only ovoid admitting \( \text{PSL}_2(7) \) acting transitively. \( \square \)

### 3.2 The symplectic polar space

**The symplectic line and regular spreads of \( W_{d-1}(q) \)**

A spread of a finite polar space \( \Sigma \) is said to be regular if it contains the regulus through any three of its generators. One way to construct a regular spread \( \mathcal{M} \), is to apply field reduction to the symplectic line \( W_1(q^{d/2}) \), thought of as an ovoid of itself. Hence \( \text{PSL}_2(q^2) \) acts 2-transitively on \( \mathcal{M} \).

**The elliptic quadric of \( W_2(q) \)**

It was proved by Thas in [Tha72], that any ovoid of \( W_2(q) \) is an ovoid of \( \text{PG}(3,q) \), and for \( q \) even, an ovoid of \( \text{PG}(3,q) \) is an ovoid of some \( W_2(q) \). Moreover, \( W_2(q) \) has an ovoid if and only if \( q \) is even, which coincides with the fact that \( W_2(q) \) is self-dual if and only if \( q \) is even (see [PT84, 3.2.1]). Let \( q \) be even and let \( O \) be the dual of a regular spread of \( W_2(q) \). Then \( O \) is an ovoid called an \textit{elliptic quadric} of \( W_2(q) \). Clearly \( \text{PSL}_2(q^2) \) acts 2-transitively on \( O \).

**The Suzuki-Tits ovoid and the Lüneburg spread of \( W_2(q) \)**

Tits [Tit62] showed that \( W_2(q) \) admits a polarity if and only if \( q = 2^{2a+1} \) for some integer \( a \). Furthermore, the absolute points of such a polarity form an ovoid \( O_a \) of \( \text{PG}(3,q) \). It turns out that \( O_a \) is an elliptic quadric when and only when \( a = 1 \), so for larger values of \( a \), the ovoids \( O_a \) are non-classical and are known as Suzuki-Tits ovoids. For \( a > 1 \), the stabilizer of \( O_a \) in \( \text{PGL}_2(q) \) contains the Suzuki group \( Sz(q) \) acting 2-transitively on \( O_a \).

Using the Klein Correspondence, it was shown by Thas [Tha72] that for \( q \) even, each spread of \( W_2(q) \) corresponds to an ovoid of \( \text{PG}(3,q) \) and conversely. The spreads corresponding to the Suzuki-Tits ovoids are the Lüneburg spreads first discovered by Lüneburg [Lü65]. Via this correspondence, the Suzuki group \( Sz(q) \) acts 2-transitively on the corresponding Lüneburg spread.

**The Herig spread of \( W_2(3) \)**

First we give a construction of Herig's spread which was first published in [Her70]. Let \( V \) be a vector space of dimension 6 over \( \text{GF}(3) \). Define an alternating form on \( V \) by

\[
(u,v) = u_1v_6 - u_6v_1 + u_2v_5 - u_5v_2 + u_3v_4 - u_4v_3.
\]

Note that \( \ell_{oo} = \{(x_1,x_2,x_3,0,0,0) : x_1,x_2,x_3 \in \text{GF}(3) \} \) is a totally isotropic line with respect to this alternating form. Let \( G \) be the group generated by the matrices

\[
\begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and define \( \mathcal{S} \) to be the orbits of \( \ell_{oo} \) under the natural action of \( G \). The set of lines \( \mathcal{S} \) define a symplectic spread of \( W_2(3) \) and the group \( G \) is 2-transitive on \( \mathcal{S} \) and is isomorphic to \( \text{SL}_2(13) \). In fact, the permutation group induced by the action of \( G \) on \( \mathcal{S} \) is \( \text{PSL}_2(13) \). Dempwolff in [Demp94] has shown that up to isomorphism, the only nonregular spreads of \( W_2(3) \) are that of Hering and those obtained via one of Albert's Twisted Fields [see [BKLS94] for more details].

**Uniqueness and non-existence results**

Much of what is given here can be found in Thas' survey [Tha01]. Hamilton and Mathon (see [HM01, Corollary 1]) proved that there are no \( m \)-systems in \( W_{d-1}(q) \) for \( m < d-4 \). So in particular, there are no ovoids in \( W_{d-1}(q) \) for \( d > 4 \). They also classified all \( m \)-systems of \( W_{d-1}(2) \) for \( d \leq 10 \) (see [HM01, 3.2]). Of these \( m \)-systems, the only transitive ones are the regular spreads and the elliptic quadrics of \( W_2(2) \). For \( W_2(3) \), Dempwolff [Demp94] has shown that the only transitive spreads of \( W_2(3) \) are the Hering spread and the regular spread. By using a computer, we have established uniqueness for 1-systems (with an insoluble stabiliser) in this space:

**Lemma 3.2.** There is a unique transitive 1-system in \( W_2(3) \) with insoluble stabiliser, which can be obtained by field reduction of a Hermitian curve \( H_2(3^2) \).
Proof. Here we can use the computer package GAP [GAP06]. Such a 1-system admits the Sylow 7-subgroup of $PSL_3(3)$, which has 520 orbits of size 7 on lines, 132 of which are partial 1-systems. Of these 132, just 15 of these are 1-systems. Three of these admit two $PSL_3(3)$, and are precisely the field reduction of a $H_3(3^2)$. Another three are field reductions of a Singer type ovoid of $H_3(3^2)$ and admit $Z_{25} : Z_5$. The remaining nine 1-systems are not transitive. □

B. Bagchi and N. N. Sastry [BS87] have shown that the only transitive $m$-systems of $W_2(q)$ are the elliptic quadric, the regular spread, the Suzuki-Tits ovoid, and the Lüneburg spread. By using techniques similar to ours, A. Cossidente and O. King reproved this result in [CK01].

3.3 The elliptic quadric
From hermitian $m$-systems

A Baer involution $\tau$ of $PTO_2(q)$ (acting on $Q_2^*(q)$) interchanges the two classes of planes on $Q_2^*(q)$. Under the Klein correspondence, one obtains a unitary polarity of $PG_3(q^2)$ defining $H_3(q^2)$. By counting the number of points of $Q_2^*(q)$ (the fixed points of $\tau$), it follows that $H_3(q^2)$ is the point-line dual quadrangle of $Q_2^*(q)$. So ovoids (resp. spreads) of $H_3(q^2)$ are equivalent to spreads (resp. ovoids) of $Q_2^*(q)$. See also [PT84, §3.2].

Uniqueness and non-existence results

Hamilton and Mathon [HM01, Corollary 1] proved that there are no $m$-systems in $Q_{d-1}^*(q)$ for $m < d/4 - 1$. So in particular, there are no ovoids in $Q_{d-1}^*(q)$ for $d > 4$. By duality, the transitive spreads of $Q_2^*(q)$ are precisely the duals of transitive ovoids of $H_3(q^2)$. Cossidente and Korchmáros [CK03] proved that for $q$ even, the only linearly transitive ovoids of $H_3(q^2)$ are the classical ones and the Singer type ovoid (see §3.1). By a result of Luyckx and Thas [LT02, LT05b], there is a unique 1-system of $Q_2^*(q)$ (for all $q$); namely that arising by field reduction of the elliptic quadric $Q_2^*(q^2)$.

3.4 The parabolic quadric
From symplectic $m$-systems

For $q$ even, $Q_2(q)$ is isomorphic to $W_{d-1}(q)$ (see [PT84]). Hence for $q$ even, we have complete information on the known transitive $m$-systems of $Q_2(q)$ by considering $m$-systems of $W_{d-1}(q)$. We also know for general $q$, that $W_2(q)$ is dual to $Q_2(q)$. Hence in $Q_2(q)$, we have complete knowledge of the transitive $m$-systems when first treating the symplectic case.

Ovoids of $Q_2(q)$ in characteristic 3

By a theorem of Thas [Tha81], $Q_2(q)$ has ovoids if and only if the Split Cayley hexagon $H(q)$ has ovoids. It is conjectured (see [OT95]) that ovoids of $Q_2(q)$ only exist for $q$ a power of 3. Also, by the natural embedding of $Q_2(q)$ in $Q_2^*(q)$, every ovoid of $Q_2(q)$ is an ovoid of $Q_2^*(q)$ (see Section 2.2).

Transitive ovoids, spreads, and $m$-systems

The Patterson ovoid
Up to isometry, there is a unique ovoid of $Q_2(q)$, first found by Patterson [Pat76]. This ovoid corresponds to an ovoid of $Q_2^*(q)$, which is unique up to semi-isometry. The Weyl group of $E_7$, which is isomorphic to $Sp_6(2)$, acts 2-transitively on the Patterson ovoid. We give a construction here that was given by Shult [Shu89]. Consider $V = GF(3^2)$ equipped with the quadratic form $Q(x_1, \ldots, x_7) = x_1^2 + \cdots + x_7^2$, which thus defines a $Q_2(q)$. Define a Fano plane structure on $\{1, 2, 3, 4, 5, 6, 7\}$ with lines the orbit of $\{1, 2, 4\}$ under the cycle $(1, 2, \ldots, 7)$. Let $G$ be a copy of $PSL_2(2)$ acting as automorphisms on this structure. Then the union of the orbits $(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0)^T$ gives an ovoid of $Q_2(q)$.

The Ree-Tits ovoids

Tits proved in [Tir95] that a polarity of $H(q)$ exists if and only if $q = 3^{2h+1}$ for some non-negative integer $h$. Moreover, he proved that the absolute points of such a polarity form an ovoid of $H(q)$. By Thas' theorem [Tha81], there arises a corresponding ovoid of $Q_2(q)$ known as the Ree-Tits ovoid. The Ree-Tits ovoid of $Q_2(q)$ admits the Ree group $G_2^*(q)$ acting 2-transitively.

The Thas-Kantor ovoids

Consider a $Q_2(q)$ subspace of $Q_2(q)$ and let $\mathcal{F}$ be the set of all lines of $H(q)$ that lie in $Q_2(q)$. It turns out that $\mathcal{F}$ is a spread of $H(q)$, and so if $q = 3$, there arises an ovoid of $H(q)$ by the duality of $H(q)$. Therefore, by Thas' theorem we have an ovoid of $Q_2(q)$. These ovoids form the family of Thas-Kantor ovoids of $Q_2(q)$ (see [Kan82a] and [Tha80]) which admit $PSU_3(q^2)$ acting 2-transitively.

Triality in $Q_2^*(q)$

Recall from above, that an ovoid of $Q_2(q)$ is an ovoid of $Q_2^*(q)$. The triality automorphism (see Section 2.3 for more details) maps an ovoid to a spread, and conversely. We will see in the following, that spreads of $Q_2^*(q)$ give rise to spreads of $Q_2(q)$, and hence applying triality to ovoids of $Q_2(q)$ is an important construction.

Spreads of $Q_2^*(q)$ and $Q_2(q)$ are equivalent

It is known (see [Luy02, p. 164]) that spreads of $Q_2(q)$ are equivalent to spreads of $Q_2^*(q)$. If $Q_2^*(q)$ has a spread $\mathcal{F}$ and $U$ is a hyperplane meeting $Q_2^*(q)$ in a $Q_2(q)$, then

$$\mathcal{F} := \{U \cap X : X \in \mathcal{F}\}$$

is a spread of $Q_2(q)$. Conversely, every plane of $Q_2(q)$ lies on a unique Greek solid and a unique Latin solid of $Q_2^*(q)$. So if $\mathcal{F}$ is a spread of $Q_2(q)$, then there exists a spread $\mathcal{F}'$ of one of the two classes of solids of $Q_2^*(q)$. Now, there is a unique ovoid $\mathcal{O}$ of $Q_2(q)$, which gives the unique ovoid of $Q_2^*(q)$ by its natural embedding. Hence, all the known spreads of $Q_2(q)$ arise from the spreads of $Q_2^*(q)$ obtained by intersection of $Q_2(q)$ with the two classes of spreads arising from applying the triality of $Q_2^*(q)$ to $\mathcal{O}$.

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3 A spread of a generalised hexagon $H$ of girth $2m$, is a set of mutually opposite lines of $H$ such that every element of $H$ is at most distance $m$ from some line of the spread.
Spreads of $Q_5(q)$

By the natural embedding of $Q_5(q)$ in $Q_6(q)$, every spread of $Q_5(q)$ is a 1-system of $Q_6(q)$. In the case of $Q_5(3)$, we know that every spread is the dual of an ovoid of $H_3(3^2)$. There is only one transitive ovoid of $H_3(3^2)$; the classical ovoid $H_3(3^2)$, which has $PSU_2(3^2)$ acting 2-transitively.

1-systems from the Hexagon

It was recognised by Shult and Thas [ST95], that by the natural embedding of the hexagon $\mathcal{H}(q)$ in $Q_6(q)$, every spread of $\mathcal{H}(q)$ is a 1-system of $Q_6(q)$. We consider the case $q = 3$ as a useful example. There are up to isometry, exactly three spreads of $\mathcal{H}(3)$ (see [DWHSV05]). They are the classical Hermitian spread (with group $G_2(2)$), the Ree-Tits spread (with group $Pt_2(2)$), and a spread recently found in [DWHSV05], which we denote $\mathcal{F}$, that has automorphism group $G = 2^3:\text{PSL}_3(2)$ (which is not a split extension). Given a 1-system of $Q_6(3)$, there is a method, often called derivation (see Section 2.4), which produces another 1-system of $Q_6(3)$. It is dual to that of hyperbolic line replacement of ovoids of $H_3(3^2)$. So given a 1-system $M$ of $Q_6(q)$ and a regulus $\mathcal{H}$ contained in $M$, by taking the opposite regulus $\mathcal{H}'$, we get a new 1-system $(M \setminus \mathcal{H}) \cup \mathcal{H}'$. Now the spread $\mathcal{F}$ consists of 7 reguli, and by replacing all of them by their opposite, we get a 1-system of $Q_6(3)$ inequivalent to $\mathcal{F}$, and not contained in any hexagon of $Q_6(3)$. A copy of $\mathcal{G}$ acts transitively on this derived 1-system.

Here we give a construction of 1-systems of $Q_6(3)$ which admit an imprimitive linear group isomorphic to $2^3:\text{PSL}_3(2)$. As before (see "The Patterson ovoid" above), we consider $GF(3^3)$ equipped with quadratic form $Q(x_1,\ldots,x_7) = x_1^2 + \cdots + x_7^2$, we define a Fano plane structure on $\{1,2,\ldots,7\}$ with lines the orbit of $\{1,2,4\}$ under the cycle $(12\cdots7)$, and we let $G$ be a copy of $\text{PSL}_3(2)$ acting as automorphisms on this structure. Then the orbits of the lines

\[ [(1,1,1,0,0,0,0),(0,1,1,0,0,1,0)] \]

and

\[ [(1,1,1,0,0,0,0),(0,1,1,0,0,1,0)] \]

under $G$ give rise to the two imprimitive transitive 1-systems of $Q_6(3)$.

Uniqueness and non-existence results

Recall from above that the ovoids and spreads of $Q_6(3)$ are known by a result of Patterson. We complete the classification of $m$-systems in this space below.

Lemma 3.3. The transitive 1-systems of $Q_6(3)$ with an insoluble stabiliser are:

(i) the three 1-systems obtained from the three spreads of the hexagon $\mathcal{H}(3)$, and

(ii) the imprimitive 1-system obtained by derivation of the imprimitive spread of $\mathcal{H}(3)$.

Proof. A transitive 1-system of $Q_6(3)$ admits the Sylow 7-subgroup $S$ of $\text{PSL}_3(3)$, which has 520 orbits on lines, each of size 7. One can then find all possible orbits on lines of size 28 which together form a 1-system. By using the software package GAP [GAP06], it was found that there are 239 1-systems admitting $S$. Of these 1-systems, 21 of them have an insoluble stabiliser, all acting transitively on the given 1-system. Then it follows by inspection that the transitive 1-systems of $Q_6(3)$ with an insoluble stabiliser either come from one of the three spreads of the hexagon $\mathcal{H}(3)$, or are equivalent to the derived imprimitive spread admitting $2^3:\text{PSL}_3(2)$.

Recall that for $q$ an odd power of 3, the Ree group $G_2(q)$ acts 2-transitively on the Ree-Tits ovoid of $Q_6(q)$. In this situation, we also have that $\mathcal{H}(q)$ admits a polarity and hence $G_2(q)$ also stabilises a spread of $\mathcal{H}(q)$; and hence a 1-system of $Q_6(q)$. The $\mathcal{H}(q)$-planes incident with the points of the Ree-Tits ovoid form a spread of $Q_6(q)$ preserved by $G_2(q)$. We show now that these are the only $m$-systems admitted by $G_2(q)$.

Lemma 3.4. The Ree group $G_2(q)$, in its action on $Q_6(q)$, has a unique orbit on totally singular points, a unique orbit on totally singular lines, and a unique orbit on totally singular planes; each of length $q^3 + 1$. Hence it acts transitively on a unique ovoid, a unique 1-system, and a unique spread of $Q_6(q)$. Moreover, the Borel subgroup of $G_2(q)$ fixes a unique line of $Q_6(q)$ on its fixed singular point; and this line is singular.

Proof. It is sufficient to show that the subgroup $B$ of the BN-pair of $G_2(q)$ fixes a unique totally singular line, a unique totally singular plane, and a unique totally singular plane of $Q_6(q)$. By [KL88a, Lemma 6], since $G_2(q)$ acts absolutely irreducibly on $GF(q^2)$, there is a unique fixed point $X$ which is an element of the Ree-Tits ovoid preserved by our copy of $G_2(q)$. To show that there is a unique fixed totally singular line, we need only to consider the lines through the unique fixed point. In the paper of Bäärnhielm [Bäa, pp. 3], there appear five types of matrices which generate $G_2(q)$. The first four generate $B$ (see [Bäa, Proposition 2.1]), and they stabilise the vector $e_1 = (0,0,0,0,0,0,1)$, which corresponds to our unique fixed point $X$. Now to see how $B$ acts on the lines through $e_1$ we consider the matrices obtained by deleting the bottom row and right column of these four types of matrices. That is, we consider the quotient geometry over $X$. One can see immediately that $B$ fixes $(e_0,e_1)$, where $e_0 = (0,0,0,0,0,0,1)$, and that this line is an element of the Ree-Tits spread of $\mathcal{H}(q)$. By inspecting the eigenspaces of these matrices, we see that there are no other fixed lines on $e_1$. Therefore $B$ fixes a unique totally singular line of $Q_6(q)$.

Consider $Q_6(q)$ as a hyperplane section of $Q_5^2(q)$. By [KL88b, Lemmata], there exists a triality $\tau$ of $Q_5^2(q)$ centralised by $G_2(q)$. So the solid $X'$ of $Q_5^2(q)$ is fixed by $B$, and this solid meets $Q_6(q)$ in a $B$-invariant plane. Without loss of generality, let us assume that $X'$ is a greek solid. Now suppose we have a plane $\pi$ of $Q_6(q)$ on $X$ fixed by $B$. Then there is a unique greek solid $Z$ of $Q_5^2(q)$ containing $Q_6(q)$, and hence a point $Y$ of $Q_5^2(q)$ fixed by $B$ in the $\tau$-orbit of $X$. If $Y$ resides in $Q_6(q)$, then $Y$ is equal to $X$ (by uniqueness) and $\pi$ is the unique plane of $Q_6(q)$ contained in $X'$. However, if $Y$ does not lie in $Q_6(q)$, then $B$ fixes the line $m$ on $X$ and $Y$. However, the perp in $Q_5^2(q)$ of $m$ meets $Q_6(q)$ in a $Q_5^2(q)$ cone with vertex $X$, whose perp is a non-singular plane meeting the 6-space spanned by $Q_6(q)$ in a non-singular line $m'$. This line $m'$ is incident with $X$ and fixed by $B$; thus contradicting the uniqueness of the line $e_1$. (Note: We will be using a similar argument in the proof of Lemma 3.13.) Hence $B$ fixes a unique totally singular plane of $Q_6(q)$.
The following lemma relates to the action of $\text{PSU}_3(q^2)$ on $Q_6(q)$. Now it is well known (see for example [Kle88a]) that $\text{PSU}_3(q^2)$ has an irreducible action on $Q_6(q)$ (resp. $Q_7^r(q)$) if and only if $q$ is a power of $3$ (resp. $q \equiv 2 \pmod{3}$). However, this is not to say that $\text{PSU}_3(q^2)$ has only irreducible actions on $Q_6(q)$ (resp. $Q_7^r(q)$) for these values of $q$. It follows from [Kle88b, Proposition 2.2] that there is just one class of reducible copies of $\text{PSU}_3(q^2)$ in $\text{PSL}_3^*(q)$, and for characteristic $3$, there is a triality of $Q_7^r(q)$ which conjugates a reducible $\text{PSU}_3(q^2)$ to an irreducible $\text{PSU}_3(q^2)$. It is important to point out here the work of Luyckx and Thas [LT05a] on $m$-systems in $Q_7^r(q)$. Their paper contains a host of useful lemmas, and they show that equivalence of locally hermitian $1$-systems is preserved by triality in $Q_7^r(q).

**Lemma 3.5.** Let $q$ be a power of $3$, and suppose that $\text{PSU}_3(q^2)$ stabilises an $m$-system $\mathcal{M}$ of $Q_6(q)$ and acts irreducibly. Then $\mathcal{M}$ is equivalent to a Thas-Kantor spread, or the spread induced by considering the $\mathcal{M}$-planes of points of this ovoid.

**Proof.** By [Kle88a, Lemma 9], the Borel subgroup $B$ of $S = \text{PSU}_3(q^2)$ fixes a unique point of $Q_6(q)$, and this point is a singular point of our given quadric $Q_6(q)$. It follows then that $S$ fixes a unique ovoid, namely a Thas-Kantor ovoid. It can be seen from the representation given by Kantor in [Kan82a, pp. 1199] that $B$ fixes a unique plane, and this plane is singular. Hence $S$ fixes a unique spread, namely that induced by taking the $\mathcal{M}$-planes incident with points of the Thas-Kantor ovoid. Now a $1$-system of $Q_6(q)$ stabilised by $S = \text{PSU}_3(q^2)$ gives rise to a $1$-system $\mathcal{M}$ of $Q_6^r(q)$, by natural embedding. By [Kle88b], there exists a triality which maps $S$ to a copy of $S$ acting irreducibly on $Q_7^r(q)$. Moreover, this triality maps $\mathcal{M}$ to a $1$-system of $Q_7^r(q)$. However, we will see in the Natural-characteristic case of the proof of Theorem 4.7 that this situation does not occur. Hence there is no $1$-system of $Q_6(q)$ stabilised by $\text{PSU}_3(q^2)$. □

It is known that there are no ovoids of $Q_6(5)$ (see for example OT95). However, spreads of $Q_6(5)$ exist since $Q_7^r(5)$ has spreads (see the next subsection). For candidate groups that we will be interested in later, we will need the following lemma.

**Lemma 3.6.** There are no transitive $m$-systems of $Q_6(5)$ admitting $\text{Sp}_2(2)$ or $\text{PSU}_3(3^2)$. Moreover, $\text{PSU}_3(3^2)$ does not stabilise a subset of points of $Q_6(5)$ of size $126$.

**Proof.** Both $\text{Sp}_2(2)$ and $\text{PSU}_3(3^2)$ contain $\text{PSL}_3(7)$, which contain two conjugacy classes of $\text{PGL}_2(5)$ (which can be readily checked by using GAP [GAP06] or Magma [Magma]). Let $G_1$ and $G_2$ be representatives for these two conjugacy classes (respectively). Since $Q_6(5)$ does not contain an ovoid, it suffices to examine the orbits of $G_1$ and $G_2$. By totally isotropic lines and planes, it turns out that neither group has a union of orbits that is a $1$-system or spread.

By [CCN85], there is a unique $7$-dimensional representation, up to equivalence, of $\text{PSU}_3(3^2)$ in characteristic $0$. It is generated by the following two matrices in $\text{GL}_7(5)$:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 4 & 4 \\
1 & 0 & 0 & 0 & 1 & 4 & 0 \\
0 & 0 & 4 & 0 & 4 & 4 & 0 \\
0 & 1 & 4 & 0 & 4 & 4 & 0 \\
1 & 0 & 4 & 0 & 4 & 4 & 0 \\
0 & 1 & 0 & 4 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 4
\end{pmatrix}

\begin{pmatrix}
0 & 4 & 0 & 4 & 4 & 4 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 4 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 4 & 0 & 0 \\
1 & 0 & 4 & 0 & 4 & 0 & 0 \\
0 & 1 & 4 & 0 & 4 & 0 & 0
\end{pmatrix}
\]

Transitive ovoids, spreads, and $m$-systems

In this representation, we can compute the orbits of $\text{PSU}_3(3^2)$ on the points of $PG_6(5)$. The orbit sizes (counting multiplicities) are

\[
\]

Clearly, $\text{PSU}_3(3^2)$ does not stabilise a set of points of size $126$. □

**3.5 The hyperbolic quadric**

The smallest case: $Q_7^r(q)$

In rank $2$, the hyperbolic quadric is a thin generalised quadrangle consisting of two classes of lines which each form one of the two spread of $Q_7^r(q)$. Since every point of $Q_7^r(q)$ is on precisely two lines, it is common to think of $Q_7^r(q)$ as a grid. It is simple to construct an ovoid of $Q_7^r(q)$, since they are simply subsets of points of size $q + 1$ which pairwise lie on different lines; for example, a conic section of $Q_7^r(q)$.

The Klein Correspondence and $Q_7^r(q)$

By the Klein Correspondence, an ovoid of $Q_7^r(q)$ corresponds to a spread of $PG_6(q)$. Moreover, the regular spreads of $PG_6(q)$ give rise to elliptic quadrics of $Q_7^r(q)$, and vice-versa. Suppose $\mathcal{S}$ is a spread of $PG_6(q)$ and let $\mathcal{H}$ be a regular of $\mathcal{S}$. If $\mathcal{H}$ is the opposite regulus of $\mathcal{H}$, then

\[
(\mathcal{S} \setminus \mathcal{H}) \cup \mathcal{H}
\]

is a spread of $PG_6(q)$. This way of creating spreads from old ones is often called derivation or regulus reversal (see Section 2.4). The derivation of a regular spread is called a Hall spread, after Marshall Hall [Hal43]. Under the Klein Correspondence, a Hall spread gives rise to an ovoid of $Q_7^r(q)$ obtained by taking an elliptic quadric $\mathcal{C}$ and a conic $\mathcal{C}$ of $\mathcal{C}$, and then replacing $\mathcal{C}$ with its opposite conic; that is, if $\pi$ is the plane defined by $\mathcal{C}$, then the polar plane $\pi^\perp$ intersects $\mathcal{C}$ in a conic; the opposite conic of $\mathcal{C}$. For $q = 3$, it was shown by Bruck [Bru69] that every spread of $PG_6(q)$ is either regular or a Hall spread. Via the Andre/Bruck-Bose construction, the Hall spread corresponds to the nearfield plane of order $9$ (see Dem97, §5.2). We also know in this case that both spreads are transitive and hence we obtain two transitive ovoids of $Q_7^r(3)$; the elliptic quadric admitting $\text{PSL}_2(9)$ (acting $2$-transitively) and the Hall ovoid admitting $2^4S_5$ (acting imprimitively).

Embeddings of quadrics

Recall from Section 2.2 that if we have an $m$-system of $Q_{2m}(q)$, then by natural embedding, we obtain an $m$-system of $Q_7^r(q)$. An important example will be the embedding $Q_7^r(q)$ of $Q_{2m}(q)$, where the Thas-Kantor ovoids ($q = 3^h$) and Ree-Tits ovoids ($q = 3^{2h+1}$) give rise to ovoids of $Q_7^r(q)$ admitting $\text{PSU}_3(q^2)$ and $G_2(q)$ respectively.

---

4 A generalised quadrangle is called thin if it has at most two lines on every point or at most two points on every line.
Triality in $Q_7^2(q)$

Recall from Section 2.3 that a triality automorphism of $Q^2_7(q)$ preserves the set of $m$-systems of $Q^2_7(q)$ with $m \in \{0, 1, 3\}$. Note however that the image of a 1-system of $Q^2_7(q)$ under a triality is a 1-system, which may be inequivalent to the original 1-system. Luyckx and Thas [LT05] showed that equivalence of locally hermitian 1-systems is preserved by triality in $Q^2_7(q)$.

Primitive ovoids

By using the O'Nan-Scott Theorem and the Classification of Finite Simple Groups, Athula Gunawardena classified the primitive ovoids of $Q^2_7(q)$. This work superseded Kleidman's classification of the 2-transitive ovoids in this space (although his classification covers other polar spaces). We describe each primitive ovoid below.

The Dye ovoid of $Q^2_7(2)$

There is a unique ovoid of $Q^2_7(2)$ and it admits $A_8$ acting 2-transitively. This ovoid is well-known and was shown to be unique by Dye [Dye77]. Cameron and Scidel [CS77] showed that for $q = 2$, inequivalent Kerdock sets yield inequivalent codes. Therefore, the uniqueness of the ovoid of $Q^2_7(q)$ can also be attributed to their work together with Kantor's construction of new Kerdock sets [Kan82a, Kan82b, Kan82c]. Let $V$ be a 9-dimensional vector space over $GF(2)$, equipped with the symmetric bilinear form $f$ defined by

$$f((a_1, \ldots, a_9), (b_1, \ldots, b_9)) = \sum_{i=1}^9 a_i b_i.$$

Consider the natural action of $S_9$ on $V$ (which permutes coordinates) and let $W$ by the set of vectors of $V$ whose coordinates sum to 0; that is, the orthogonal complement of $(1, 1, \ldots, 1)$ with respect to $f$. Clearly $W$ is an 8-dimensional subspace of $V$ invariant under $S_9$. Moreover, $W$ yields an absolutely irreducible representation of $S_9$ over $GF(2)$ known commonly as the fully deleted permutation module of $S_9$ (see [KL90, §§3]). The restriction of $f$ to $W$ induces a non-degenerate symmetric bilinear form on $W$ forming a $Q^2_7(2)$ polar space. Let $\mathcal{C}$ be the set of elements of $W$ of weight 8; that is the orbit of $(1, 1, 1, 1, 1, 1, 1, 1, 0)$ under $S_9$. Clearly $\mathcal{C}$ is a set of 9 singular points of $W$ which constitute an ovoid of $Q^2_7(2)$. It is not difficult to see that $A_8$ acts 2-transitively on $\mathcal{C}$.

The unique ovoid of $Q^2_7(3)$

Recall that the Patterson ovoid of $Q_8(3)$ is the unique ovoid of this parabolic quadric. By embedding, this ovoid gives rise to an ovoid $\mathcal{C}$ of $Q^2_7(3)$. It was shown by Patterson [Pat76] that $\mathcal{C}$ is the unique ovoid of $Q^2_7(3)$ up to semi-isometry (see also [Shu89]).

Kantor's unitary ovoids of $Q^2_7(q)$, $q \equiv 2 \pmod{3}$

In 1982, Kantor [Kan82a, Kan82b] constructed an infinite family of transitive ovoids of $Q^2_7(q)$ for $q \equiv 2 \pmod{3}$. Let $q$ be a prime power and suppose that $q \equiv 2 \pmod{3}$. Let $V$ be the Lie algebra $\mathfrak{sl}_2(q^2)$ of $3 \times 3$ hermitian matrices over $GF(q^2)$. That is, each element $m$ of $V$ is a matrix satisfying $m^T = m$, where $m^T$ is the transpose matrix of $m$. Clearly $GU_3(q^2)$ acts on $V$ in its adjoint action:

$$m^T = gm(g^*)^T.$$

Let Trace be the ordinary trace map on matrices which computes the sum of the diagonal entries of a matrix. It turns out that since $q \equiv 2 \pmod{3}$, the Killing form $(m, n) \rightarrow \text{Trace}(mn)$ on $V$ is a non-degenerate symmetric bilinear form of $+1$ type. Hence we can identify $V$ with $Q^2_7(q)$.

Now $GU_3(q^2)$ has an irreducible module over $K = GF(q^2)$, which is the twisted tensor product $K^n \otimes (K^2)^{t}$ obtained by multiplying column vectors $(x_1, x_2, x_3)$ of $K^3$ by row vectors $(\alpha^T, \beta^T, \gamma^T)$ of $(K^2)^{t}$. This twisted tensor product contains a subset $\mathcal{U}_q$ of $\mathfrak{sl}_2(q^2)$ consisting of elements of the form

$$\begin{pmatrix} a & b & c \\ d & \gamma & \alpha \\ e & f & \beta \end{pmatrix} = \begin{pmatrix} a^{1+q} & ab & ac \\ a & a^* & \alpha \\ ab & b^{1+q} & b\gamma \end{pmatrix}.$$

Clearly these elements are singular under the Killing form (as they have zero trace) and $GU_3(q^2)$ stabilises $\mathcal{U}_q$ in its action on $V$, and induces $PGU_3(q^2)$ acting 2-transitively. It is also not difficult to see that a pair of elements of $\mathcal{U}_q$ are opposite and therefore it is an ovoid of $Q^2_7(q)$.

The Kantor desarguesian ovoids of $Q^2_7(q)$

Also in Kantor's important work of 1982 (see [Kan82a, Kan82b]), there is a construction of an infinite family of irreducible transitive ovoids of $Q^2_7(q)$ for $q$ even. Recall that if one applies field reduction of the symplectic line $W_1(q^2)$, one obtains a regular spread $\mathcal{S}$ of $W_1(q)$ admitting $SL_2(q^2)$ acting 3-transitively. For $q$ even, we have that $W_1(q)$ is isomorphic to $Q_8(q)$ and hence $\mathcal{S}$ is also a spread of $Q_8(q)$. Recall from the previous section, that there is a spread $\mathcal{S}'$ of $Q^2_7(q)$ such that $\mathcal{S}'$ is the set of hyperplane intersections of $\mathcal{S}$. Note that $\mathcal{S}'$ admits a reducible group isomorphic to $SL_2(q^2)$ acting 3-transitively, so under the triality automorphism, we obtain an ovoid $\mathcal{U}_q$ of $Q^2_7(q)$ admitting an absolutely irreducible group isomorphic to $SL_2(q^2)$ (see [Kle87]). Hence, for $q$ even, we have a transitive ovoid $\mathcal{U}_q$ admitting $SL_2(q^2)$.

Remark 3.7. We could also define $\mathcal{U}_q$ in much the same way as Kantor's unitary ovoid. By Steinberg's Twisted Tensor Product Theorem, the minimal representation of $SL_2(q^2)$ in natural characteristic (of degree 2 over $GF(q^2)$) gives rise to an absolutely irreducible representation $U : SL_2(q^2) \rightarrow T \otimes T^{*} \otimes T^{*}$. Since $U^*$ is equivalent to $U$, it follows that $U$ can be written over $GF(q)$. Thus $U$ is an absolutely irreducible representation of degree 8 over $GF(q)$. Now let $\mathcal{C}$ be the diagonal of this representation:

$$(a, b) \otimes (a^T, b) \otimes (a^T, b^T) : a, b \in GF(q^2).$$

It turns out that one can equip the Lie Algebra $\mathfrak{sl}_2(q^2)$ with a non-degenerate symmetric bilinear form, which under $U$, gives rise to an embedding of $\mathcal{C}$ in $Q^2_7(q)$. Furthermore, we have that $\mathcal{C}$ is an ovoid of $Q^2_7(q)$ admitting $PSL_2(q^2)$.

One could also define $\mathcal{C}$ via the octonions thought of as a composition algebra of index 4. That is, we model the octonions as $4 \times 4$ matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
where $\alpha, \beta \in \text{GF}(q)$ and $a, b \in \text{GF}(q)^3$. Under the usual entry-wise scalar multiplication and addition, these matrices form a vector space, and there is also a multiplication (see § 5 of Buekenhout and Cohen’s chapter of [Bue95]) which produces a nonassociative algebra over \text{GF}(q). There is a quadratic form defined by

$$Q(x) = \alpha \beta + a \cdot b,$$

where $a \cdot b$ is the scalar product of $a$ and $b$, such that $Q$ is multiplicative; i.e., $Q(xy) = Q(x)Q(y)$. Therefore, these matrices together with this nondegenerate quadratic form (of Witt index 4) produce a “split Cayley composition algebra”. This setting provides a model of $\text{Q}_7^+(q)$ and we can define $\mathcal{O}$ in the following. For all $t \in \text{GF}(q)^3$, let $t^2$ denote the element of $\text{GF}(q)^3$ obtained by squaring each coordinate of $t$. Then for $q$ even, the set given by

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & t \\ t^2 & t \cdot t^2 \end{pmatrix} : t \in \text{GF}(q)^3 \right\}$$

is equivalent to the Kantor desarguesian ovoid.

2-systems of $\text{Q}_7^+(q)$ all arise from spreads

If $\mathcal{M}$ is a 2-system of $\text{Q}_7^+(q)$ and $\mathcal{S}$ is a given family of generators, then the set of all generators in $\mathcal{S}$ containing an element of $\mathcal{M}$ forms a spread of $\text{Q}_7^+(q)$. The converse is also true by a result of Shult and Thas [ST94, Theorem 11], which we recast below:

**Theorem 3.8 (Shult and Thas, 1994).** Let $S_1$ and $S_2$ be spreads of $\text{Q}_7^+(q)$, where the generators of $S_1$ and the generators of $S_2$ belong to different families. Then for each $s \in S_1$, there is exactly one $s' \in S_2$ such that $s \cap s'$ is a plane. All such planes form a 2-system of $\text{Q}_7^+(q)$.

1-systems of $\text{Q}_7^+(q)$

Let $q$ be odd and consider the rank 1 polar space $\text{Q}_7(q)$ thought of as an ovoid of itself. Then by field reduction, we obtain a 1-system of $\text{Q}_7^+(q)$ (see Section 2.1). It turns out, that for $q$ odd, this is the unique 1-system (up to equivalence) of $\text{Q}_7^+(q)$. In general, the 1-systems of $\text{Q}_7^+(q)$ were classified by Shult and Thas’s seminal work [ST94]. We revisit their result below:

**Theorem 3.9 (Shult and Thas, 1994).** For $q$ odd, there is a unique 1-system (up to equivalence) of $\text{Q}_7^+(q)$. For $q$ even, every 1-system of $\text{Q}_7^+(q)$ comes from a spread of an embedded $\text{Q}_4(q)$.

Recall from the previous section that the transitive spreads of $\text{Q}_4(q)$ are the duals of the elliptic quadric and Suzuki-Tits ovoid of $W_3(q)$.

Uniqueness and non-existence results

Recall that there is a unique ovoid of $\text{Q}_7^+(3)$, that is, the Patterson ovoid. Hence by triality, the spreads of $\text{Q}_7^+(3)$ are known, and by Theorem 3.8, the 2-systems are also known in this space. From exploration by computer, there is a unique 1-system of $\text{Q}_7^+(3)$ admitting an insoluble transitive group that is not an embedded 1-system of $\text{Q}_6(q)$.

**Lemma 3.10.** There is a unique insoluble transitive 1-system of $\text{Q}_7^+(3)$ that is not a 1-system of $\text{Q}_6(q)$. It is constructed as follows: Let $\mathcal{S}$ be the 1-system of $\text{Q}_6(q)$ obtained by deriving the exceptional spread of the hexagon $\mathcal{H}(3)$. Let $\mathcal{S}'$ be the 1-system of $\text{Q}_7^+(3)$ obtained by the natural embedding of $\text{Q}_6(q)$. Under the triality automorphism of $\text{Q}_7^+(3)$, we obtain another 1-system of $\text{Q}_7^+(3)$ that is incompatible to $\mathcal{S}'$.

**Proof.** A 1-system of $\text{Q}_7^+(3)$ consists of 28 lines, and so the stabiliser of a transitive 1-system contains a Sylow 7-subgroup $\mathcal{S}$ of $\text{PGO}_7^+(3)$ (since $\mathcal{S}$ has size $7$). Every orbit of $\mathcal{S}$ on the lines of $\text{Q}_7^+(3)$ has size $7$, and so it suffices to enumerate all unions of four of these orbits, where each orbit is itself a partial 1-system. Of the 5200 orbits of size 7, just 796 of them are partial 1-systems. By using the software package GAP [GAP06], it was found that there are 211 1-systems admitting $\mathcal{S}$. Of these 1-systems, 145 of them have an insoluble stabiliser, each acting transitively on the given 1-system. The result then follows by inspection of these 1-systems.

Similarly, there are only three ovoids of $\text{Q}_7^+(5)$ up to equivalence, and this was established recently by Charns and Dempwolff [CD01]. These ovoids are the Conway-Kleinman-Wilson ovoid, the Cooperstein ovoid, and Kantor’s unitary ovoid of $\text{Q}_7^+(5)$. Only the latter two are transitive. We will need the following two lemmas in deducing the transitive 1-systems of $\text{Q}_7^+(5)$.

**Lemma 3.11.** There is one conjugacy class of absolutely irreducible subgroups of $\text{PTO}_7^+(5)$ isomorphic to $A_7$, and they do not act on a 1-system of $\text{Q}_7^+(5)$.

**Proof.** According to [Kle87], an absolutely irreducible $A_7$ in $\text{PGO}_7^+(5)$ is a subgroup of $\text{PSU}_3(5)$ acting on the Lie algebra of hermitian matrices of trace 0. By computer, one can compute that there is no union of the orbits of $A_7$ on lines that yields a 1-system.

**Lemma 3.12.** There is no 1-system of $\text{Q}_7^+(5)$ admitting $\text{PSL}_2(8)$ or $\text{Sp}_4(2)$.

**Proof.** If $\text{PSL}_2(8)$ occurred as the stabiliser of a 1-system $\mathcal{M}$, then the stabiliser of a pair of lines of $\mathcal{M}$ in $\text{PSL}_2(8)$ would be an elementary abelian group of order 8 with all involutions conjugate in $\text{PSU}_3(5)$. The only elementary abelian subgroups of order 8 of the Sylow 2-subgroup of the stabiliser of a pair of opposite lines in $\text{PGO}_7^+(5)$ have involutions with 72 fixed points. In $\text{PSL}_2(8)$, there is an involution normalising a cyclic subgroup of order 7. In $\text{PGO}_7^+(5)$, none of the involutions normalising a cyclic subgroup of order 7 have 72 fixed points. Hence $\text{PSL}_2(8)$ does not arise. Now we consider the simple group $\text{Sp}_4(2)$, which has a unique conjugacy class of subgroups of index $5^2 + 1$ and they are each isomorphic to $2^5 : A_6$. The nonabelian composition factors of the stabiliser of a totally singular line are just two copies of $A_7$. Therefore $2^5 : A_6$ cannot be the subgroup of the stabiliser of a line. So this case does not arise.

The most interesting case for us will be when we have either $\text{PSU}_3(q^2)$ or $2G_2(q)$ acting. Let $S$ be one of these two simple groups and let $B$ be the Borel subgroup of $S$. That is, $B$ is the normaliser of a Sylow $p$-subgroup of $S$. It is also true (see [Kle88a, pp. 124]) that $B$ is the stabiliser in $S$ of a totally singular point on its corresponding ovoid (i.e., the Thas-Kantor or Ree-Tits ovoid). Now $\text{PSU}_3(q^2)$ has an absolutely irreducible action on $\text{Q}_7^+(q)$ if and only if
q \equiv 2 \pmod{3}$, and moreover, its Borel subgroup fixes a unique point in the corresponding module [Kle88a, Lemma 9]. The group $2G_2(q)$ does not have an irreducible 8-dimensional representation in characteristic 3, but it does have a unique absolutely irreducible action on $Q_8(q)$ (see the proof of [Kle88b, Proposition 4.2.4]). Likewise, $PSU_3(q^2)$ has an absolutely irreducible action on $Q_8(q)$ if and only if $q$ is a power of 3, and this action is unique (c.f., [Kle88b, Proposition 2.2] and [Kle88a, Lemma 8]). So it will be the action of $S$ on $Q_8(q)$ that will be important in our analysis of $m$-systems of $Q_8(q)$. If $S$ acts absolutely irreducibly on $Q_8(q)$, then $B$ fixes a unique point and this point is singular.

**Lemma 3.13.** Let $q$ be a power of 3 and suppose $S$ is $PSU_3(q^2)$ or $2G_2(q)$ acting (reducibly) on $Q_8(q)$. Let $B$ be the Borel subgroup of $S$, that is, the stabiliser of a point $X$ on the corresponding invariant ovoid of $Q_8(q)$. Then $B$ does not fix any totally singular line of $Q_8(q)$ incident with $X$.

**Proof.** First, let $U$ be a non-degenerate hyperplane of $Q_8(q)$ fixed by $S$, and let $C$ be a Thas-Kantor ovoid of $U$ if $S$ is $PSU_3(q^2)$, or a Rees-Tits ovoid of $U$ when $S$ is $2G_2(q)$. Let $X$ be a totally singular point of $C$ and let $\ell$ be a totally singular line of $Q_8(q)$ incident with $X$, but not contained in $U$. (So $\ell$ meets $U$ in $X$.) Suppose $B$ fixes $X$. Then $\ell^\perp$ is a $Q_8^-(q)$ cone with vertex $\ell$ and $\ell$ meets the totally singular subspaces of $U$ in a $Q_8^-(q)$ cone with vertex $\ell$. Now $(\ell^\perp \cap U)^\perp$ is a non-singular plane and it meets $U$ in a non-singular line $m$ tangent to the incipient $Q_8(q)$. Note that $m$ is stabilised by $B$, and $B$ and $C$ have reduced the problem to a question about the action of $B$ on $Q_8(q)$. We will show that such a line $m$ does not exist.

First suppose that $S$ is $PSU_3(q^2)$. Here we reconstruct the irreducible action of $S$ on $Q_8(q)$ given by Kantor [Kan82a]. For all $a \in GF(q^2)$, define $\bar{a} = a^\alpha$, $T(a) = a + \bar{a}$, and $N(a) = a\bar{a}$. Let $V$ be the Lie algebra of matrices of the form

$$M = \begin{pmatrix} a & b & c \\ \gamma & \bar{a} & \bar{b} \\ \bar{b} & \gamma & \bar{a} \end{pmatrix}$$

with $a, b, c, \gamma \in GF(q^2)$, $a, b, c \in GF(q)$, and $a + \alpha + \bar{a} = 0$. Define a quadratic form $Q$ on $V$ by $Q(M) = a^2 + \alpha \bar{a} + \bar{a}^2 + T(b, \gamma) + bc$. Here, we have that the radical of $V$ is $\langle \ell \rangle$, and the quotient space is $Q_8(q)$. Our group $S$ acts naturally on $V$ by conjugation. The induced bilinear form is the Killing form $\langle M, N \rangle = \text{Trace}(MN)$. The Thas-Kantor ovoid is simply the projection of the set of one-dimensional subspaces of $V$ given by $(\langle \ell \rangle : Z^\perp = 0)$. Without loss of generality, we may suppose that $X$ has representative

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Kantor shows that the Sylow 3-subgroup of $B$ consists of all matrices of the form

$$M[\bar{\lambda}, \mu] = \begin{pmatrix} 1 & -\mu & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix}, \quad T(\lambda) + N(\mu) = 0.$$

Suppose $m$ is the line joining $X$ and $Y$, where

$$Y = \begin{pmatrix} a & b & c \\ \gamma & \bar{a} & \bar{b} \\ \bar{b} & \gamma & \bar{a} \end{pmatrix}, \quad a + \alpha + \bar{a} = 0.$$

Transitive ovoids, spreads, and $m$-systems

So the image of $Y$ under $M[\bar{\lambda}, \mu]$ is

$$\begin{pmatrix} 1 & \bar{\lambda} & \bar{\mu} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ \gamma & \bar{a} & \bar{b} \\ \bar{b} & \gamma & \bar{a} \end{pmatrix} = \begin{pmatrix} a - \mu \bar{a} & b - \mu \bar{b} & c - \mu \bar{c} \\ \gamma - \alpha \bar{a} + \beta \bar{b} & \bar{a} - \beta \bar{b} & \bar{c} - \beta \bar{c} \\ \bar{b} - \beta \bar{b} & \bar{a} - \beta \bar{b} & \bar{c} - \beta \bar{c} \end{pmatrix}.$$

Let $\bar{\lambda}, \mu \neq 0$. Since $M[\bar{\lambda}, \mu]$ fixes $m$, we have that $Y' = Y$ is on the line $(X, Y)$ and therefore

$$Y' = Y - \begin{pmatrix} \mu \gamma + \lambda \bar{b} - a \bar{a} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} \end{pmatrix},$$

from which it immediately follows that either $\mu = 0$ or $Y' - Y \in (X)$. The latter case cannot occur as then $\mu = \gamma = 0$. So $\mu = 0$, which implies that

$$Y' = Y - \begin{pmatrix} \mu \gamma + \lambda \bar{b} - a \bar{a} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} \end{pmatrix},$$

and hence $\gamma = 0$. Therefore

$$Y' = Y - \begin{pmatrix} 0 & \alpha \bar{a} + \mu \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} + \gamma \mu + \lambda \bar{b} \end{pmatrix}.$$

So we have furthermore that $T(\beta \mu) \neq 0$ and $\alpha = a = 0$. Thus we have

$$Y = \begin{pmatrix} 0 & \beta \bar{c} \\ 0 & 0 \end{pmatrix},$$

and so every vector on $m$ is of the form

$$\begin{pmatrix} 0 & \beta \bar{c} \\ 0 & 0 \end{pmatrix},$$

for some $d, t \in GF(q)$. However, every one of these vectors is totally singular, a contradiction. Therefore, $B$ does not fix any totally singular line of $Q_8(q)$ incident with $X$.

For the case $S = 2G_2(q)$, note that by Lemma 3.4 we have that the Borel subgroup $B$ fixes a unique line on its fixed point of the Rees-Tits ovoid, and this line is singular. Hence the existence of a fixed tangent line $m$ is impossible, and hence $B$ does not fix any totally singular line of $Q_8(q)$ incident with $X$. 

It is conjectured that there are no ovoids of $Q_{2n+1}(q)$ for $r \geq 4$. This was shown to be true for $q$ in characteristic $2$ and $3$ by Blokhuis and Moorhouse [BM92]. It is well known that there are no spreads of $Q_{2n+1}(q)$ for $n \geq 1$. The reason for this being that two generators of $Q_{2n+1}(q)$ that are in the same family are never disjoint.
4 The classification

There is a common situation that arises in the work of this paper, and in facilitating the proofs in this section, we make the following observations. Suppose we have a map $\beta$ from a finite classical polar space $\mathcal{P}$ onto a finite classical polar space $\mathcal{P}'$ that preserves incidence and maps $m$-systems to $m'$-systems, such as a map induced by field reduction (see Section 2.1) or a natural embedding of polar spaces with equal ovoid numbers (see Section 2.2). Assume also that we have a group $J$ of isometries of $\mathcal{P}$, and that there is a unique conjugacy class of subgroups of the full isometry group of $\mathcal{P}$ isomorphic to $J$. Then if $J$ has a unique orbit of size $\mu(\mathcal{P})$ and $\mathcal{M}$ is an $m$-system of $\mathcal{P}$, it follows that every $m'$-system of $\mathcal{P}'$ admitting $J$ is equivalent to $\beta(\mathcal{M})$. With this observation in mind, we list the following situations for which this occurs.

**Lemma 4.1.** For the pairs of groups $J$ and isometry groups $I$ listed in the table below, there is a unique conjugacy class of subgroups of $I$ isomorphic to $J$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Sz}(q^{d/4})$</td>
<td>$\text{P}I\text{Sp}_d(q)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^2)$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
<tr>
<td>$\text{PSL}_2(q^{d/2})$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
<tr>
<td>$\text{PSL}_2(q^{d/2})$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^2)$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^{d/2})$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^2)$</td>
<td>$\text{SP}_{10}(q^2)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^{d/2})$</td>
<td>$\text{SP}_{10}(q^2)$</td>
</tr>
<tr>
<td>$\text{PSU}_3(q^2), q = 3^k$</td>
<td>$\text{PS}_6(q)$</td>
</tr>
</tbody>
</table>

**Proof.** By a result of Lüneburg (see [Lü1, 27.3 Theorem] or [Lü2]), there is a unique conjugacy class of subgroups of $\text{P}GL_d(q^{d/4})$ isomorphic to $\text{Sz}(q^{d/4})$. This implies that there is a unique conjugacy class of subgroups of $\text{Gl}_d(q^{d/4})$ isomorphic to $\text{Sz}(q^{d/4})$. Now $\text{Sz}(q^{d/4})$ is absolutely irreducible and so by Schur's Lemma, we have that $\text{C}_{\text{Gl}_d(q^{d/4})}(J)$ consists of scalar matrices. It then follows that there is a unique conjugacy class of subgroups of $\text{GSp}_d(q^{d/4})$ isomorphic to $\text{Sz}(q^{d/4})$. Now by [KL1, Proposition 4.3.10], there is a unique conjugacy class of subgroups of $\Gamma\text{Sp}_d(q^{d/4})$ isomorphic to $\text{GSp}_d(q^{d/4})$. Therefore, there is a unique conjugacy class of subgroups of $\Gamma\text{Sp}_d(q^{d/4})$ isomorphic to $\text{Sz}(q^{d/4})$.

The proofs of the remaining inclusions are readily found in [KL1]. We provide a table below which lists the relevant results.

<table>
<thead>
<tr>
<th>Case Reference</th>
<th>Case Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2) Proposition 4.3.7</td>
<td>(6) Proposition 4.3.6</td>
</tr>
<tr>
<td>(3) Proposition 4.3.10</td>
<td>(7) Proposition 4.1.4</td>
</tr>
<tr>
<td>(4) Proposition 4.3.16</td>
<td>(8) Proposition 4.3.18</td>
</tr>
</tbody>
</table>

**Lemma 4.2.** Let $J$ be an isometry group from a row of the table below. If $\mathcal{M}$ is an $m$-system of a classical polar space $\mathcal{P}$ and $\mathcal{M}$ admits $J$, then $\mathcal{M}$ is equivalent to one of the $m$-systems associated to $J$ in the table below:

<table>
<thead>
<tr>
<th>Case</th>
<th>$J$</th>
<th>$m$-systems</th>
<th>Possible $m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\text{W}_d(q^{d/4})$, $q = 2k+1$</td>
<td>$\text{Sz}(q^{d/4})$</td>
<td>Field reduction of Suzuki-Tits ovoid</td>
<td>$d - 4$</td>
</tr>
<tr>
<td>2. $\text{W}_d(q)$</td>
<td>$\text{PS}_6(q^2)$</td>
<td>Field reduction of Hermitian curve $H_2(q^2)$</td>
<td>1</td>
</tr>
<tr>
<td>3. $\text{W}_d(q)$</td>
<td>$\text{PS}_6(q^{d/2})$</td>
<td>Regular spread</td>
<td>$d - 1$</td>
</tr>
<tr>
<td>4. $\text{Q}_d(q)$</td>
<td>$\text{PS}_6(q^{d/2})$</td>
<td>Field reduction of Elliptic quadric $Q_d(q^{d/2})$</td>
<td>$d - 4$</td>
</tr>
<tr>
<td>5. $\text{Q}_d(q)$</td>
<td>$\text{PS}_6(q^2)$</td>
<td>Field reduction of Hermitian curve $H_2(q^2)$</td>
<td>1</td>
</tr>
<tr>
<td>6. $\text{H}_d(q^2)$</td>
<td>$\text{PS}_6(q^{d/2})$</td>
<td>Field reduction of Hermitian curve $H_2(q^2)$</td>
<td>1</td>
</tr>
<tr>
<td>7. $\text{H}_d(q^2), q = 3^k$</td>
<td>$\text{PS}_6(q^2)$</td>
<td>Hermitian curve $H_2(q^2)$</td>
<td>0</td>
</tr>
<tr>
<td>8. $\text{Q}_d(q^2), q = 3^k$</td>
<td>$\text{PS}_6(q^2)$</td>
<td>Field reduction of Thas-Kantor ovoid, Hermitian $1$-system, or spread of $Q_4(q)$</td>
<td>1, 3, 5</td>
</tr>
</tbody>
</table>

**Proof.** For each of the cases above, we will prove the following three conditions:

(i) If $J$ acts faithfully on $\Omega$, then there is a unique subset of $\Omega$ of size $|\mathcal{M}|$ stabilised by $J$.

(ii) There is just one conjugacy class of subgroups of the collineations of $\mathcal{P}$ isomorphic to $J$.

(iii) There exists a known $m$-system of $\mathcal{P}$ admitting $J$.

We will also show that $m$ takes on a specific value. In the cases where field reduction is involved, we use the following argument. Let $\varphi$ be the natural vector space isomorphism from $V_d(q)$ to $V_d(q)$, and suppose there is an $m'$-system $\mathcal{M}$ admitting $J$, of a polar space with underlying set $V_d(q)$. Let $\alpha$ and $\beta$ be two points of $\Omega$. Then the image $\alpha J = \alpha J \beta$ of $\alpha$ under $\varphi$ is a $d/b$-space of $V_d(q)$, so consider the stabiliser of $\alpha J = \alpha J \beta$ which we identify with $J_{\alpha J \beta}$. We will now look at the subspaces of $V_d(q)$ invariant under $J_{\alpha J \beta}$. Let $v$ be a vector lying in the subspace $\alpha J = \alpha J \beta$ (the image of $\beta$ under $\varphi$). In the cases that we will be employing this argument, $J$ acts 2-transitively on the elements of $\mathcal{M}$, and so we know that

$$[J : J_{\alpha J \beta}] = q^{d/2} - 1,$$

where $d + 1$ is the size of $\mathcal{M}$. If $\alpha J \beta$ is trivial, then we will have

$$|\mathcal{M}| = q^{d/2}(d - 1),$$

where $d$ is some positive integer. This will be true of all the groups $J$ we will be considering. Since $J_{\alpha J \beta}$-invariant subspaces are a union of $J_{\alpha J \beta}$-orbits on vectors of $V_d(q)$, and such subspaces have size a power of a prime, it follows that the only nontrivial subspaces of $V_d(q)$ left invariant under $J_{\alpha J \beta}$ are $\alpha J$ and $V_d(q)$ itself. Therefore $\mathcal{M}$ will be equivalent to the field reduction of $\mathcal{P}$, which is a $(d/b - 1)$-system of $\mathcal{P}$.

It will then follow that $\mathcal{M}$ is equivalent to the known $m$-system given by (iii) above. In each case below, we have condition (ii) by Lemma 4.1. All that remains is to prove that the first and third conditions stand.
(1) In the seminal paper of Suzuki [Su62, §15], it was shown that $S\bar{z}(q^{d/4})$ has a unique conjugacy class of subgroups of index $|A|$ (which equals $q^{d/2} + 1$) and this is the minimum non-trivial degree of $S\bar{z}(q^{d/4})$. Therefore, condition (i) is satisfied. The final condition will be shown to be valid by considering the Suzuki-Tits ovoid and Lüneburg spread of $W_1(q^{d/4})$, for which $S\bar{z}(q^{d/4})$ acts $2$-transitively, and the image of it under field reduction to $W_{d-1}(q)$. Here we can apply the generic argument given above since $J$ acts $2$-transitively on the Suzuki-Tits ovoid and Lüneburg spread of $W_1(q)$. To adapt the argument, note that $b = 4$ and that $J$ is a Zassenhaus group, and hence $J_{\varphi}$ is trivial. Therefore

$$|\varphi| = q^{d/2} |\varphi| = q^{d/2}(q^{d/4} - 1)$$

which is certainly not the power of a prime. Therefore $A$ is equivalent either to the image of the Suzuki-Tits ovoid (under $\varphi$), which is a $(d/4 - 1)$-system of $W_{d-1}(q)$, or to the image of the Lüneburg spread, which is a $(d/2 - 1)$-system of $W_{d-1}(q)$.

(2) It is well known (see [Coo78]) that if $q \neq 5$, then $PSU_2(q^2)$ has a unique conjugacy class of subgroups of index $q^3 + 1$, and that these subgroups are point stabilisers in the nontrivial minimum degree permutation representation of $PSU_2(q^2)$. If $q = 5$, then the two minimum degrees of $PSU_2(q^2)$ are 50 and 126 = $q^3 + 1$. Hence (i) is satisfied. Moreover, if one takes the Hermitian curve $H_3(q^2)$, thought of as an ovoid of itself, and consider the image of it under field reduction, we obtain a 1-system of $W_8(q)$ admitting $PSU_2(q^2)$. We show now that $m = 1$, and thus that $A$ is equivalent to the classical 1-system. Let $S = PSU_2(q^2)$, and let $\alpha$ be a point of $H_3(q^2)$. Here we can apply the generic argument given above since $J$ acts $2$-transitively on the Hermitian curve $H_3(q^2)$ of $H_3(q^2)$. To adapt the argument, note that $b = 2, d = 6$, and the stabiliser of two points in $J$ has order $(q^2 - 1)/gcd(3, q^2 + 1)$.

Thus

$$|\varphi| = q^3 |\varphi| = q^3(q^2 - 1)/gcd(3, q^2 + 1)$$

which is certainly not the power of a prime. Therefore $A$ is equivalent to the image of the Hermitian curve $H_3(q^2)$ (under $\varphi$), which is a 1-system of $W_8(q)$.

(3) It is a classical result, but can also be found in the more recent work of Cooperstein [Coo81] that $PSL_2(q^{d/2})$, (where $d > 2$) has a unique conjugacy class of subgroups of index $q^{d/2} + 1$. Except for $q^{d/2} = 9$, we have that these subgroups are point stabilisers in the nontrivial minimum degree permutation representation of $PSL_2(q^{d/2})$. Now if $q^{d/2} = 9$, that is $q = 3$ and $d = 4$, then it is not difficult to calculate by computer that $PSU_2(q^2)$ has two orbits on lines; the regular spread and its complement. Moreover, if one takes the symplectic line $W_1(q^{d/2})$, thought of as an ovoid of itself, and the image of it under field reduction, we obtain a regular spread of $W_{d-1}(q)$ admitting $PSU_2(q^{d/2})$. We show now that $m = d/2 - 1$, that is, $A$ is the regular spread of $W_{d-1}(q)$. Again we can apply the generic argument given above since $J$ acts $2$-transitively on the symplectic line $W_1(q^{d/2})$. We adapt the argument by noting that $b = 2$ and that $J$ is a Zassenhaus group, and hence $J_{\varphi}$ is trivial. Therefore

$$|\varphi| = q^{d/2} |\varphi| = q^{d/2}(q^{d/4} - 1)/gcd(2, q^{d/2} - 1)$$

which is certainly not the power of a prime. Therefore $A$ is equivalent to the image of $W_1(q^{d/2})$ (under $\varphi$), which is a regular spread of $W_{d-1}(q)$.

(4) By similar arguments to that above we see that in this case, $A$ is equivalent to a $(d/4 - 1)$-system of $Q_{d-1}(q)$ obtained by field reduction of the Elliptic Quadric $Q_3(q^{d/4})$ thought of as an ovoid of itself.

(5) Here we refer to the arguments in (2) to establish that condition (i) is satisfied. Now the dual of the Hermitian ovoid of $H_3(q^2)$ is a spread of $Q_{3}(q^2)$ admitting $PSU_2(q^2)$ (hence (iii) is satisfied). So $A$ is equivalent to this spread.

(6) Again, by similar arguments to that above we see that $A$ is equivalent to an $(a - 1)$-system of $H_3(q^2)$ obtained by field reduction of the Hermitian curve $H_3(q^2)$ thought of as an ovoid of itself.

(7) Field reduction is not involved in this case, but again, we refer to the arguments in (2) to establish that condition (i) is satisfied. Every non-degenerate hyperplane section is an ovoid of $H_3(q^2)$ admitting $PSU_2(q^2)$ (hence (iii) is satisfied). So $A$ is a classical ovoid of $H_3(q^2)$.

(8) The size of $A$ is $q^2 + 1 = (q^2 + 1)$. So by the arguments in (2), condition (i) is satisfied. Existence of $m$-systems of $Q_3(q^2)$ admitting $J$ follow from field reduction of the Thas-Kantor ovoid, Hermitian 1-system, and the spread arising from the Thas-Kantor ovoid (see the "Note" above and (7)). Hence by the arguments at the beginning of this proof on field reduction, we have that $A$ is equivalent to one of the aforementioned $m$-systems.

We start out the classification of insoluble transitive $m$-systems of finite classical polar spaces in the realm of the Hermitian varieties.

**Theorem 4.3 (The Hermitian Varieties).** Let $q$ be a power of a prime $p$ and let $d$ be a positive integer greater than $1$. If $A$ is an $m$-system of $H_3(q^2)$ admitting an insoluble transitive group, then $A$ is either the rank 1 polar space $H_3(q^2)$ thought of as an ovoid of itself, the exceptional ovoid of $H_3(q^2)$, or a classical ovoid of $H_3(q^2)$.

**Proof.** Suppose $q = p^e$ for some positive integer $e$ and suppose $A$ is an $m$-system of $H_3(q^2)$ such that a group $G$ of collineations acts transitively on $A$. Let $H$ be the stabiliser of an element of $A$. We may assume that $d > 3$, as the case $d = 3$ is when $A$ is the rank 1 polar space $H_3(q^2)$ thought of as an ovoid of itself. Now the ovoid number is $p^{d/2} = q^2 + 1$, where $d$ is $d$ or $d - 1$ according to whether $d$ is odd or even respectively. If $(p, e) = (2, 3)$, then $e = 3, f = 1$, and $d = 4$. In this case, we have that $\Phi_{2d}(p) = 1$, hence we cannot apply the Theorems of [BP]. However, Brouwer and Wilbrink [BW90] have shown that the only transitive $m$-systems of $H_3(q^2)$ are classical ovoids. So assume now that $(p, e) \neq (2, 3)$, that is, $\Phi_{2d}(p) > 1$. Let $G = G \cap PGU_2(q^2)$ and $H = H \cap PGU_2(q^2)$. Now $\Phi_{2d}(p)$ divides $p^{d/2} + 1$, which in turn divides $|G|$. So $\Phi_{2d}(p)$ divides $|G|$, from which it follows that $\Phi_{2d}(p)$ divides $|G|$. Hence we can apply [BP, Theorem 1] or [BP, Theorem 2] to $G$.

First we look at the case that $d$ is odd. By [BP, Theorem 1] we have that $PSU_2(q^2) \leq G$. Now $PSU_2(q^2)$ is not contained in $H$ as $H$ is corefree. Note that $PSU_2(q^2) : PSU_2(q^2) \cap H$ divides $|G| : |H|$ as $PSU_2(q^2)$ is a normal subgroup of $G$. Now $|G| : |H|$ divides $p^{d/2} + 1$, and by [Coo78], we have that the minimum degree of $PSU_2(q^2)$ for $d$ an odd integer greater than 3 is $(q^2 + 1)(q^{d/2} - 1)/gcd(2, q^{d/2} - 1)$. Note that this figure is clearly larger than the ovoid number $q^2 + 1$ as $d > 4$; a contradiction. Hence, this case does not arise.

Now suppose $d$ is even. By [BP, Theorem 2] we have one of the following:
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arising by field reduction of an ovoid of \( H_3(q^2) \), or can be obtained via field reduction of one of the above.

Proof. Suppose \( q = p^f \) for some positive integer \( f \) and suppose \( \mathcal{M} \) is an m-system of \( W_{d-1}(q) \) such that a group \( G \) of collineations acts transitively on \( \mathcal{M} \). Let \( H \) be the stabilizer of an element of \( \mathcal{M} \).

First assume that \( q = 2 \) and \( d = 6 \). The m-systems of \( W_5(2) \) were classified by Hamilton and Mathon, and only the regular spread of \( W_5(2) \) is transitive (see [HM01, 3.2]). Now assume that \( d = 2 \). Then we have that \( \mathcal{M} \) is the entire rank 1 polar space considered as an ovoid of itself. As noted in the previous section, the case \( d = 4 \) was settled by Coster and King [CK01] where it was found that \( \mathcal{M} \) is the elliptic quadric, regular spread, Suzuki-Tits ovoid, or the L"{u}neburg spread of \( W_3(q) \). So suppose \( d \leq 6 \) and \( d \neq (2, 6) \). Let \( G = G \cap PGSp_{2d}(q^2) \) and \( H = H \cap PGSp_{2d}(q^2) \). Now \( |G : H| = |\mathcal{M}| \times x \) where \( x = |G : PGSp_{2d}(q^2) : H : PGSp_{2d}(q^2)| \). Since \( x \) divides \( f \), we can apply [BP, Corollary 5.3] to \( G \) with the assumption that \( d > 4 \).

We have two cases: the "Extension field" examples and the "Nearly simple" examples.

For the "Extension field" case, we have that \( \mathcal{M} \) admits a nonabelian simple group \( S \) and one of three situations occurs: (a) \( S \cong PSU_2(2^{d/2}) \), (b) \( S \cong S_6(2^{d/2}) \) and \( q \) is even, or (c) \( S \cong PSU_3(q^2) \) with \( d = 6 \) and \( q \) odd. By Lemma 4.2, it follows that \( \mathcal{M} \) is an m-system obtained by field reduction respectively of (a) the symplectic line \( W_1(q^2) \) (and \( m = d/2 - 2 \)), (b) a Suzuki-Tits ovoid or L"{u}neburg spread of \( W_3(q^{d/2}) \) (and \( m \in \{d - 1, d/2 - 1\} \)), or (c) an ovoid of \( H_3(q^2) \) (and \( m = 1 \)).

In the "Nearly simple" case we have that \( S \leq H \leq Aut(S) \) when \( S \) is a nonabelian simple group. Since we have assumed \( d > 4 \), the "Alternating Group case" and "Natural-characteristic case" do not arise. For the "Cross-characteristic case" we have that \( q = p \) (and hence \( G = \tilde{G} \)) and the table below lists the possibilities for this case.

<table>
<thead>
<tr>
<th>( S )</th>
<th>( d )</th>
<th>( q )</th>
<th>( S \cap H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSU_3(2^2)</td>
<td>6</td>
<td>4 - A_4, 4^2 - A_5</td>
<td></td>
</tr>
<tr>
<td>PSU_2(7)</td>
<td>6</td>
<td>3</td>
<td>S_3</td>
</tr>
<tr>
<td>PSU_2(13)</td>
<td>6</td>
<td>3</td>
<td>13 : 3</td>
</tr>
<tr>
<td>PSU_2(25)</td>
<td>12</td>
<td>2</td>
<td>S_6, S_5</td>
</tr>
</tbody>
</table>

By Proposition 3.2, the only transitive 1-system of \( W_5(3) \) is the classical one. Dempewolff [Dem94] has shown that the only transitive spreads of \( W_5(3) \) are the Hering spread and the regular spread. So we can assume that \( q \neq 3 \).

By using MAGMA [Mag96], it was found that \( PSU_3(3^2) \) (as a subgroup of \( PGSp_6(q) \)) has no orbit on 126 totally isotropic points. Therefore the case \( S = PSU_3(q^2) \) does not arise. From inspecting [JLPW95], the only irreducible 12-dimensional characteristic 2 representation of \( SL_2(25) \) is obtained by considering the action of \( SL_2(25) \) on the fully deleted permutation module, which preserves a direct sum decomposition of this module into two 12-dimensional modules. By using GAP, it was found that \( SL_2(25) \) has orbits on points of \( W_{11}(2) \) of lengths 65, 325, 325, 650, 780, and 1950. Since an m-system of \( W_{11}(2) \) covers \( 65 \times 2^{1+1-1} - 1 \) points, it follows that \( SL_2(25) \) does not act on an m-system of \( W_{11}(2) \).

\[ \square \]
Theorem 4.5 (The Elliptic Quadrics). Let \( q \) be a power of a prime \( p \) and let \( d \) be an even positive integer greater than 2. If \( M \) is an \( m \)-system of \( Q_{d-1}(q) \) admitting a transitive group of collineations, which is not one-dimensional semilinear, then \( M \) is either the rank 1 polar space \( \mathcal{Q}_2(q) \), the classical spread of \( \mathcal{Q}_2(q) \) (the dual of a classical ovoid of \( H_0(q^2) \), the exceptional spread of \( \mathcal{Q}_3(5) \) (the dual of the exceptional ovoid of \( H_3(2^5) \)), or can be obtained via field reduction of an elliptic quadric \( \mathcal{Q}_d^*(q) \).

Proof. Suppose \( q = p^f \) for some positive integer \( f \) and suppose \( M \) is an \( m \)-system of \( Q_{d-1}(q) \) such that a group of collineations \( G \) acts transitively on \( M \). Let \( H \) be the stabiliser of an element of \( M \). Now, \( Q_2(q) \) is dual to \( H_0(q^2) \), and so we may refer to Theorem 4.3 for the case that \( d = 6 \). Assume now that \( d > 6 \). Let \( G = \mathcal{G} \cap \text{PGO}_2(q) \) and \( H = \mathcal{H} \cap \text{PGO}_2(q) \). Now, \( \mathcal{G} \cap \mathcal{H} = \{ M \} / x \) where \( x \) divides \( f \), and so we can apply [BP, Corollary 5.4] to \( G \). Since \( d > 6 \), we have only the “Extension field” examples where we have that \( M \) admits a nonabelian simple group \( S \cong \text{PSL}_2(q^{d/2}) \). By Lemma 4.2, it follows that \( M \) is an \( m \)-system obtained by field reduction of the elliptic quadric \( \mathcal{Q}_d^*(q^{d/2}) \).

Theorem 4.6 (The Parabolic Quadrics). Let \( q \) be a power of a prime \( p \) and let \( d \) be an odd integer greater than 1. If \( M \) is an \( m \)-system of \( Q_{d-1}(q) \) admitting a transitive group of collineations, which is not one-dimensional semilinear, then \( M \) is either the rank 1 polar space \( \mathcal{Q}_2(q) \), an elliptic quadric of \( \mathcal{Q}_2(q) \), a regular spread of \( \mathcal{Q}_2(q) \) (for \( q \) even), a Patterson ovoid of \( \mathcal{Q}_2(q) \), a 1-system of \( \mathcal{Q}_3(q) \) obtained from a spread of the hypergraph \( \mathcal{H}(3) \), a derivation of an imprimitive 1-system of \( \mathcal{Q}_3(q) \), a Thas-Kantor ovoid of \( \mathcal{Q}_3(q) \) (or induced spread), or a Ree-Tits ovoid, 1-system, or spread of \( \mathcal{Q}_3(q) \).

Proof. Suppose \( q = p^f \) for some positive integer \( f \) and suppose \( M \) is an \( m \)-system of \( Q_{d-1}(q) \) such that a group of collineations \( G \) acts transitively on \( M \) and \( G \) is not one-dimensional semilinear. Let \( H \) be the stabiliser of an element of \( M \). First we note that \( Q_2(q) \) is a rank 1 polar space, and so we assume that \( d > 3 \). Also, if \( q = 2 \), then \( Q_{d-1}(q) \) is isomorphic to \( W_{d-2}(q) \), and so we can refer to Theorem 4.4. Now \( Q_2(q) \) is dual to \( W_2(q) \), and so again, we may refer to Theorem 4.4 for that case. For the case that \( d = 7 \) and \( q = 3 \), we may refer to Lemma 3.3, so in total, we can assume that \( d > 5 \), is odd, and \( q, d \neq (3, 7) \). Let \( G = \mathcal{G} \cap \text{PGO}_2(q) \) and \( H = \mathcal{H} \cap \text{PGO}_2(q) \). Now, \( \mathcal{G} \cap \mathcal{H} = \{ M \} / x \) where \( x \) divides \( f \), and so we can apply [BP, Corollary 5.5] to \( G \). We have two cases: the reducible and the Nearly simple examples.

In the reducible examples case, we have that \( G \) fixes a hyperplane section \( U \) of \( \mathcal{Q}_2(q) \) of type \( \mathcal{Q}_2(q) \). Note that \( \mathcal{Q}_2(q) \) is the classical spread of \( \mathcal{Q}_2(q) \). However, a reducible \( \text{PSU}_3(q^2) \) does not have an orbit of length \( q^2 + 1 \) on totally singular planes, which we will see from the following. In the natural action of \( \text{PSU}_3(q^2) \) on the totally isotropic points of \( H_2(q^2) \), the stabiliser of two points is contained in a group of order \( (q^2 - 1) \text{gcd}(3, q + 1) \). Inside this group we have an exceptional group of order \( (q + 1) \text{gcd}(3, q + 1) \). Now, \( G \) fixes a hyperbolic line of \( H_2(q^2) \) pointwise, and by duality, we see that \( G \) fixes a regular \( \mathcal{Q}_2(q) \) line. So \( G \) preserves a decomposition of \( \mathcal{Q}_2(q) \) into an orthogonal sum \( V_1 \oplus V_2 \) where \( V_1 \) is a non-degenerate 2-space of minus type and \( V_2 \) is a non-degenerate 4-space of plus type. The only 3-dimensional \( C \)-invariant subspace of \( \mathcal{Q}_2(q) \) is then an orthogonal sum of a 2-dimensional \( C \)-invariant subspace with the \( U \) (which is non-singular). Hence, in the action of \( C \) on \( \mathcal{Q}_2(q) \) (as a subgroup of a reducible \( \text{PSU}_3(q^2) \)), there is a unique conjugacy class of such in \( \text{PTo}(q) \) there is a totally singular plane left invariant by \( C \). Hence there are no totally singular planes left invariant by \( \text{PSU}_3(q^2) \). Therefore, a reducible \( \text{PSU}_3(q^2) \) does not have an orbit of length \( q^2 + 1 \) on totally singular planes, and so we have ruled out the case that \( M \) admits \( \text{PSU}_3(q^2) \) and is not contained in \( U \).

In the Nearly simple examples case, we have that \( d = 7 \) and \( S \subseteq \mathcal{G} \leq \text{Aut}(S) \) where \( S \) is a finite nonabelian simple group. There are two subcases here; the Cross-characteristic and Natural characteristic examples. In the former, we have that \( q = 5 \) and \( S \subseteq \{ \text{PSL}_2(8), \text{Sp}_2(2), \text{PSU}_3(3^2) \} \). By Lemmas 3.6 and 3.12, there are no transitive \( m \)-systems of \( Q_6(5) \) admitting \( \text{PSL}_2(8), \text{Sp}_2(2) \) or \( \text{PSU}_3(3^2) \), and so this case does not arise. In the “Natural-characteristic”
case, we have that $p = 3$ and $G$ has a unique conjugacy class of subgroups of index $q^3 + 1$. We also have that $G$ acts 2-transitively of degree $q^3 + 1$ and we have either $S = \operatorname{PSU}_3(q^2)$ or $S = \mathfrak{P}_2(q)$ (and $q \neq 3$). By Lemma 3.4 and Lemma 3.5, we have one of the following: $S$ is $\operatorname{PSU}_3(q^2)$ and $\mathcal{M}$ is a Thas-Kantor ovoid of $Q_d(q)$ or the induced spread (by considering $K(q)$-planes on the Thas-Kantor ovoid), or $S$ is $\mathfrak{P}_2(q)$ (where $q$ is an odd power of 2) and $\mathcal{M}$ is a Ree-Tits ovoid, 1-system, or Ree-Tits spread of $Q_d(q)$.

**Theorem 4.7 (The Hyperbolic Quadrics).** Let $q$ be a power of a prime $p$ and let $d$ be an even positive integer greater than 2. If $\mathcal{M}$ is an $m$-system of $Q_{d-2}^+(q)$ admitting a transitive group of collineations, which is not one-dimensional semilinear, then one of the following occurs:

(i) $d = 4$ (rank 1) and $\mathcal{M}$ is an ovoid of $Q_4^+(q)$.

(ii) $d = 6$ and $\mathcal{M}$ is one of:

(a) an ovoid of $Q_6(q)$;

(b) a Hall ovoid of $Q_8(3)$;

(c) $q$ odd, unique 1-system;

(d) $q$ even, 1-system obtained from a spread of $Q_6(q)$.

(iii) $d = 8$ and $\mathcal{M}$ is one of:

(a) the unique ovoid of $Q_8^+(2)$;

(b) Patterson ovoid of $Q_8^+(3)$;

(c) Cooperstein ovoid of $Q_8^+(5)$;

(d) Kantor's 2-transitive ovoid of $Q_8^+(q)$ admitting $\operatorname{PSL}_2(q^3)$, and $p = 2$;

(e) Kantor's unitary 2-transitive ovoid of $Q_8^+(q)$ admitting $\operatorname{PSU}_3(q^2)$, and $q \equiv 2 \pmod{3}$;

(f) a 2-system obtained by a spread of $Q_8^+(q)$;

(g) an $m$-system obtained from an $m$-system of $Q_6(q)$;

(h) obtained by triality of one of the above.

(iv) obtained by field reduction of one of the above.

**Proof.** Suppose $q = p^f$ for some positive integer $f$ and suppose $\mathcal{M}$ is an $m$-system of $Q_{d-2}^+(q)$ such that a group of collineations $G$ acts transitively on $\mathcal{M}$ and $G$ is not one-dimensional semilinear. Let $\mathcal{N}$ be the stabiliser of an element of $\mathcal{M}$. First we assume that $d > 4$, since we can regard $Q_4^+(q)$ as a trivial case (see Section 3.5). For the case that $d = 6$, note that the Klein Correspondence between $Q_4^+(q)$ and $\mathfrak{P}_2(q)$ becomes relevant. Now by Theorem 3.9, the 1-systems of $Q_4^+(q)$ are known. Since there are no spreads of $Q_4^+(q)$, we may assume that $\mathcal{M}$ is an ovoid of $Q_4^+(q)$, which corresponds to a spread of $\mathfrak{P}_2(q)$. For $q = 2$, it is well known that every spread of $\mathfrak{P}_2(q)$ is regular, so $\mathcal{M}$ is an elliptic quadric in this case. For $q = 3$, it was shown by Bruck [Brue69] that every spread of $\mathfrak{P}_2(q)$ is either regular, or obtained by reversing a regular spread (a Hall spread). By the Klein Correspondence, the ovoids of $Q_4^+(3)$ are either elliptic quadrics, or obtained from an elliptic quadric by derivation (see Section 2.4). If $(q, d) = (2, 8)$, then $\Phi_8^+(2) = 1$ and we refer to Hamilton and Mathon's classification of the $m$-systems of $Q_8^+(2)$ (see [HM01]). They found that the only transitive $m$-systems of $Q_8^+(2)$ are the unique ovoid of $Q_8^+(2)$, its image under triality (the Desarguesian spread), the two 1-systems of $Q_8^+(2)$ (by embedding), and a unique 2-system obtained by intersection of two spreads. For $(q, d) = (3, 8)$, we can apply Lemma 3.10. Altogether, we can assume that

$$d > 4 \quad \text{and} \quad (q, d) \neq \{(2, 6), (3, 6), (2, 8), (3, 8)\}.$$  

By similar arguments to those in the proof of Theorem 4.4, $\Phi_9^+(q)$ is coprime to $f$ and so we can apply [BP, Corollary 9]. Suppose first that $G$ is reducible. That is, we have that $G$ fixes a subspace $U$ of $P_2(q)$ and $\dim(U) = m \in \{d - 1, d - 2\}$ (that is, $U$ is a hyperplane or secundum section). So by [KL90, §4.1], we have two cases:

(a) $U$ is isometric to $Q_{d-2}(q)$ and $G^U \leq \operatorname{GO}_{d-2}(q)$.

(b) $U$ is isometric to $Q_{d-3}(q)$ and $G^U \leq \operatorname{GO}_{d-3}(q)$.

**Hyperplane section case:** In the first case (a), we apply [BP, Corollary 8] to $G^U$. We have that $G^U$ is in the Nearly simple case where $S \leq G^U \leq \operatorname{Aut}(S)$, $d = 8$, and either $q = 5$ and $S \in \{\operatorname{PSL}_2(8), \operatorname{Sp}_2(2), \operatorname{PSU}_3(3^2)\}$, or

(ii) $p = 3$ and $S \in \{\operatorname{PSL}_2(q^2), \mathfrak{P}_2(q)\}$.

Suppose we are in case (i). We can assume that $\mathcal{M}$ is a 1-system as the ovoids of $Q_8^+(5)$ have been classified in [CD01]. By Lemma 3.12, there are no 1-systems of $Q_7^+(5)$ admitting $\operatorname{PSL}_2(8)$ or $\operatorname{Sp}_2(2)$. Since $G$ acts faithfully on $U$ (as it preserves an orthogonal form), and so $\mathcal{M}$ admits $S$, we have that $S = \operatorname{PSU}_3(3^2)$. The set of intersections of lines of $\mathcal{M}$ with the hyperplane $U$ gives rise to a set of $q^3 + 1$ points of $U$, which is impossible by Lemma 3.6. So we have the second case (ii) above. Now $S$ has a unique permutation representation of degree $q^3 + 1$; its canonical 2-transitive representation. Here, we can assume that $\mathcal{M}$ is an ovoid or 1-system (since spreads, and thus 2-systems, are obtained via ovoids). Suppose $\mathcal{M}$ is an ovoid. Since $S$ acts transitively on $\mathcal{M}$, we can refer to Kleidman's classification [Klee88a] and establish that $\mathcal{M}$ is induced by a Thas-Kantor ovoid or a Ree-Tits ovoid of $Q_6(q)$. So suppose that $\mathcal{M}$ is a 1-system.

If $\mathcal{M}$ is contained in $U$, then we can refer to Theorem 4.6 to establish that $\mathcal{M}$ is a Ree-Tits 1-system. So suppose that $\mathcal{M}$ is not contained in $U$ and let $\mathcal{M}$ be the set of points $\{m \cap U : m \in \mathcal{M}\}$. By 2-transitivity of $S$, either every pair of points of $\mathcal{M}$ are collinear, or every pair of points are not collinear. In the former case, we would have that $\mathcal{M}$ is contained in a plane, which is a contradiction as $\mathcal{M}$ has $q^3 + 1$ elements. Therefore $\mathcal{M}$ is an ovoid of $U$. Since this ovoid admits $S$, we have by Theorem 4.6 that $\mathcal{M}$ is a Thas-Kantor ovoid or a Ree-Tits ovoid. The points of this ovoid are each incident with a unique $K(q)$-plane, the entire collection of which form a spread of $U$. Each element of this spread is incident with a unique line of $Q_8^+(q)$ giving rise to a spread $S$ of $Q_8^+(q)$. Moreover, by this correspondence each line of $\mathcal{M}$ is contained in a unique solid of $S$. However, this situation cannot occur by Lemma 3.13 (i.e., triality must be applied to the line-solid pair to get a point-line incident pair) and therefore, no matching 1-system exists; that is, there is no 1-system in this situation for which its corresponding spread (as obtained above) of $Q_8^+(q)$ has its elements each containing a unique line of the 1-system.

To summarise, we have in the Hyperplane Section case that if $\mathcal{M}$ is an ovoid or 1-system, then it arises from an ovoid or 1-system of $Q_d(q)$.

**Secundum section case:** In case (b), we apply [BP, Corollary 5.6] to $G^U$. We have one of the following:

(i) $\operatorname{PSL}_2(q^{d-2}) \leq G^U \leq \operatorname{GO}_{d-2}(q)$.
(ii) \(q\) odd, \(d = 8\), \(\text{PSU}_3(q^2) \leq G \leq \text{GO}_q^+(q)\).

From [Co078], we have that the minimum degree of \(\text{PSL}_2(q^{d-1}/2)\) is equal to \((q^{d-1} - 1)/(q - 1)\). Note that \(\text{PSL}_2(q^{d-1}/2) \setminus \text{PSL}_2(q^{d-1}/2) \setminus H^U \neq 1\) as \(H^U\) is corefree. Now if \((q^{d-1} - 1)/(q - 1)(q^{d-1}/2 - 1)\) divides \((q^{d-1}/2 - 1)/2\), then \(G^U\) is a divisor of \([G : H] = q^{d-1/2} - 1\), which implies the contradiction. So the first case does not arise by system since the 2-transitive ovoids were determined in [Ke88a]. If \(M\) is contained in \(U\), suppose that \(M\) is induced by the Hermitian spread of \(Q_3^+(q)\). So is fixed by \(S = \text{PSU}_3(q^2)\). Such a hyperplane section exists since \(S\) acts trivially on \(U^2\), set of intersections of the elements of \(M\) with \(H^U\) give rise to a Thas-Kantor ovoid and a \(\overline{M}\), which lead to a contradiction. Therefore, \(M\) does not reside outside of the secundum section \(U\).

So summarise, we have in the Secundum Section case that if \(M\) is an ovoid or 1-system, then it arises from an ovoid or 1-system of \(Q_3^+(q)\).

Now suppose we are in the Extension field examples case. Here, we have that \(d = 14\), \(p = 3\), and \(\text{PSU}_3(q^2) \leq G\). Moreover, \(G\) preserves a field extension structure on \(Q_3^+(q)\); that reduction of a 4-system of \(Q_4^+(q^2)\); namely, a Thas-Kantor ovoid, Hermitian 1-system, or

In the "Symplectic type" examples, we have that \(q = 5\), \(d = 8\), and \(G \leq 2^{1+5} : O_6^-(2)\) which has order divisible by 126, contains \(A_5\). We may assume that \(A_5\) is absolutely irreducible, and to \(A_5\). The ovoids of \(Q_5^+(5)\) were classified by Dempwolff and Chars [CD01] (and hence by [LJPW95]), there is only conjugacy class of subgroups of \(\text{PSL}_2(q^2)\) isomorphic to \(2^{1+5} : O_6^-(2)\) which has hence by [JLPW95], there is only conjugacy class of subgroups of \(\text{PSL}_2(q^2)\) isomorphic to \(2^{1+5} : O_6^-(2)\) which has order divisible by 126, contains \(A_5\). We may assume that \(A_5\) is absolutely irreducible, and to \(A_5\). The ovoids of \(Q_5^+(5)\) were classified by Dempwolff and Chars [CD01] (and hence by [LJPW95]), there is only conjugacy class of subgroups of \(\text{PSL}_2(q^2)\) isomorphic to \(2^{1+5} : O_6^-(2)\) which has

Now suppose we are in the "Nearly simple" case. So \(d = 8\) (and hence \(q \neq 2, 3\) by assumption) and \(S = G \leq \text{Aut}(S)\) where \(S\) is a finite nonabelian simple group. We have three subcases: the "Alternating group" case, the "Cross-characteristic" case, and the "Natural characteristic" case.

**Alternating group case:** Here we have that \(q = 5\), \(n \in \{7, 9, 10\}\), \(S = A_n\), and \(G\) has a unique transitive action of degree \(q^2 + 1\). In each case, \(A_n\) acts primitively on \(M\), and so if \(M\) is an ovoid, then we can refer to the work of Gevandera [Gun00] to establish that \(M\) is the Cooperstein ovoid or Kantor’s unitary ovoid. So assume that \(M\) is a 1-system.

(1) \(n = 9, 10\): There are only two orbits of \(A_9\) acting on totally singular lines of size 126; none of which are 1-systems. This argument also rules out \(n = 10\).

(2) \(n = 7\): \(A_7\) does not act on a 1-system by Lemma 3.11.

**Cross-characteristic case:** Here \(q = 5\) and one of the following holds:

<table>
<thead>
<tr>
<th>Case</th>
<th>(S)</th>
<th>(S \cap H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) (\text{PSL}_3(4))</td>
<td>(2^4 : D_{10})</td>
<td>(2^4 : D_{10})</td>
</tr>
<tr>
<td>(2) (\text{PSL}_2(8))</td>
<td>(2^5)</td>
<td>(2^5)</td>
</tr>
<tr>
<td>(3) (\text{Sp}_4(2))</td>
<td>(2^5 : A_8)</td>
<td>(2^5 : A_8)</td>
</tr>
</tbody>
</table>

The cases (2) and (3) have been ruled out by Lemma 3.12. So we are left with Case (1) where \(S = \text{PSL}_3(4)\). Now there are two conjugacy classes of subgroups of \(S\) of index 126, and each is isomorphic to \(2^4 : D_{10}\). It turns out that this group cannot be a subgroup of the stabiliser of a totally singular line in \(\text{PGO}_5^+(5)\). This is because there is no elementary abelian group of order 16 in the stabiliser of a totally singular line in \(\text{PGO}_5^+(5)\), with all involutions conjugate in \(\text{PGO}_5^+(5)\).

**Natural-characteristic case:** Here we have that \(G\) has a unique conjugacy class of subgroups of index \(q^2 + 1\) and either \(S = \text{PSL}_2(q^2)\), or \(S = \text{PSU}_3(q^2)\).

(i) \(S = \text{PSL}_2(q^2)\): Notice that \(G\) is primitive. So if \(M\) is an ovoid, then by [Gun00], \(M\) is the ovoid constructed by Kantor. There is a unique conjugacy class of absolutely irreducible subgroups of \(\text{PSL}_2(q^2)\) isomorphic to \(\text{PSL}_2(q^2)\) (see [Ke88a] or the proof of [Kle87, Proposition 2.3.6]). We shall show now that \(M\) cannot be a 1-system. Since \(G\) acts on the aforementioned Kantor ovoid, we know that \(H\) stabilises a totally singular point. If \(M\) is a 1-system, then \(H\) must also fix a line of \(M\), as there is a unique transitive action of \(G\) of degree \(q^2 + 1\). Now \([S] \cap H\) is a divisor of \([S] = q^2 + 1\) and \(q^2 + 1\), however the stabiliser of a non-degenerate line in \(\text{PGO}_5^+(5)\) does not have \(q^2 + 1\) in its order.

(ii) \(S = \text{PSU}_3(q^2)\): By [Kle87], there is a unique conjugacy class of subgroups of \(\text{Aut} \text{PSU}_3(q^2)\) isomorphic to \(S\). Notice that \(G\) is primitive. So if \(M\) is an ovoid, then by [Gun00], \(M\) is the unitary ovoid constructed by Kantor. We shall show now that \(M\) cannot be a 1-system. By [Kan82a, pp. 1201], \(S\) has 3 orbits on singular points, namely:

(a) the Kantor unitary ovoid;
(b) a set of size \((q^2 + q)(q^3 + 1)\) (the orbit of Kantor’s \(\{Y\}\));
(c) a set of size \((q^3 + 1)\) (the orbit of Kantor’s \(\{Y'\}\)).

Now the lines of a 1-system cover a set of \((q + 1)(q^2 + 1)\) points. None of the orbits above are small enough for a 1-system to exist.

**References**


Frobenius complements of exponent dividing $2^m \cdot 9$

Enrico Jabara and Peter Mayr

(Communicated by Karl Strambach)

Abstract. We show that every group of exponent $2^m \cdot 3^n$ ($m, n \in \mathbb{N}, n \leq 2$) that acts freely on some abelian group is finite.

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1 Results

Let $V$ be a group, and let $G$ be a group of automorphisms of $V$. We say that $G$ acts freely on $V$ if $v^g \neq v$ for all $v \in V \setminus \{1\}$ and $g \in G \setminus \{1\}$. In the literature this concept is also often called regular or fixed-point-free action of $G$ on $V$.

We consider free actions of groups of finite exponent. In [1] the first author proved that groups of exponent 5 that act freely on abelian groups are finite. In the present note we show the following.

Theorem 1.1. Let $V$ be an abelian group, and let $G$ be a group of automorphisms of $V$. If $G$ has exponent $2^m \cdot 3^n$ for $0 \leq m$ and $0 \leq n \leq 2$ and $G$ acts freely on $V$, then $G$ is finite.

Every finite group that acts freely on an abelian group is isomorphic to a Frobenius complement in some finite Frobenius group (see Lemma 2.6). Let $G$ be as in Theorem 1.1. By the classification of finite Frobenius complements (see [6]) the factor of $G$ by its maximal normal 3-subgroup is isomorphic to a cyclic 2-group, a generalized quaternion group, $SL(2, 3)$, or the binary octahedral group of size 48.

Corollary 1.2. Let $F$ be a near-field whose multiplicative group has exponent $2^m \cdot 3^n$ for $0 \leq m$ and $0 \leq n \leq 2$. Then either $|F| \in \{2^3, 3^2, 5^2, 7^2, 17^2\}$ or $F$ is a finite field of prime order.

We note that there exist near-fields of orders $3^2, 5^2, 7^2, 17^2$ that are not fields. Every zero-symmetric near-ring with 1, whose elements satisfy $x^k = x$ for a fixed integer $k > 1$, is a subdirect product of near-fields satisfying the same equation (see [4] or the corresponding result for rings by Jacobson [2]). Hence, by Corollary 1.2, every zero-symmetric near-ring with 1 that satisfies $x^{m-1} = x$ for some natural number $m$ is a subdirect product

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