First Order Logic

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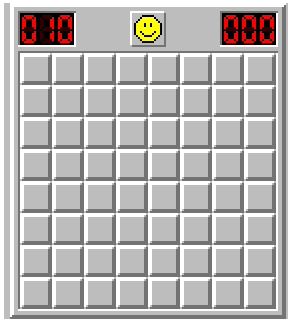
[Based on slides from Burr Settles]

Problems with propositional logic

- Consider the game "minesweeper" on a 10x10 field with only one landmine.



- How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?

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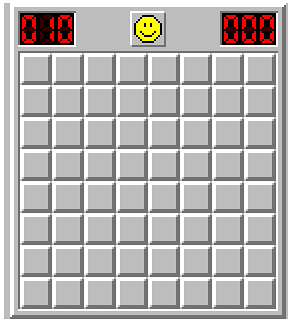
- Consider the game "minesweeper" on a 10x10 field with only one landmine.



- How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?
- Intuitively with a rule like

$$\text{landmine}(x,y) \Rightarrow \text{number1}(\text{neighbors}(x,y))$$

but propositional logic cannot do this…

Problems with propositional logic

- Propositional logic has to say, e.g. for cell (3,4):
  - Landmine_3_4 ⇒ number1_2_3
  - Landmine_3_4 ⇒ number1_2_4
  - Landmine_3_4 ⇒ number1_2_5
  - Landmine_3_4 ⇒ number1_3_3
  - Landmine_3_4 ⇒ number1_3_5
  - Landmine_3_4 ⇒ number1_4_3
  - Landmine_3_4 ⇒ number1_4_4
  - Landmine_3_4 ⇒ number1_4_5
  - And similarly for each of Landmine_1_1, Landmine_1_2, Landmine_1_3, …, Landmine_10_10!
- Difficult to express large domains concisely
- Don’t have objects and relations
- First Order Logic is a powerful upgrade

Ontological commitment

- Logics are characterized by what they consider to be ‘primitives’

Logic	Primitives	Available Knowledge
Propositional	facts	true/false/unknown
First-Order	facts, objects, relations	true/false/unknown
Temporal	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief 0…1
Fuzzy	degree of truth	degree of belief 0…1

First Order Logic (FOL) syntax

- User defines these primitives:
  - Constant symbols **(i.e. the "individuals" in the world)**: Jerry, 2, Madison, Green, …
  - Function symbols (mapping individuals to individuals): Sqrt(9), Distance(Madison, Chicago)
  - Predicate symbols (mapping from individuals to truth values) : Teacher(Jerry, you), Bigger(sqrt(2), x)
- FOL supplies these primitives:
  - Variable symbols: x, y
  - Connectives (same as PL): $\wedge \vee \neg \Rightarrow \Leftrightarrow$
  - Quantifiers $\forall$, $\exists$

“Things” in FOL

- **Term**: an object in the world
  - **Constant (i.e. the “individuals” in the world)**: Jerry, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**($term_1$, …, $term_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.

"True/False" in FOL

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read "Jerry (is)Teacher(of) you"
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - $\mathbf{term_1 = term_2}$
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object

FOL syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\wedge \vee \neg \Rightarrow \Leftrightarrow$
    - Spouse(Jerry, Jing) $\Rightarrow$ Spouse(Jing, Jerry)
    - Less(11,22) $\wedge$ Less(22,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets defines the predicates

FOL quantifiers

- Universal quantifier: $\forall$
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is $\Rightarrow$
  - Forms if-then rules
  - "all humans are mammals"

    $$\forall x \; \mathbf{human(x)} \Rightarrow \mathbf{mammal(x)}$$

  - Means if x is a human, then x is a mammal

FOL quantifiers

$$\forall x\ \mathbf{human(x)} \Rightarrow \mathbf{mammal(x)}$$

- It's a big AND: Equivalent to the conjunction of all the instantiations of variable x:

$$(\mathbf{human(Jerry)} \Rightarrow \mathbf{mammal(Jerry)}) \land$$
$$(\mathbf{human(Jing)} \Rightarrow \mathbf{mammal(Jing)}) \land$$
$$(\mathbf{human(laptop)} \Rightarrow \mathbf{mammal(laptop)}) \land \ldots$$

- Common mistake is to use $\land$ as main connective

$$\forall x\ \mathbf{human(x)} \land \mathbf{mammal(x)}$$

- This means everything is human and a mammal!

$$(\mathbf{human(Jerry)} \land \mathbf{mammal(Jerry)}) \land$$
$$(\mathbf{human(Jing)} \land \mathbf{mammal(Jing)}) \land$$
$$(\mathbf{human(laptop)} \land \mathbf{mammal(laptop)}) \land \ldots$$

FOL quantifiers

- Existential quantifier: ∃
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is ∧
  - “some humans are male”

    $$\exists x \; human(x) \wedge male(x)$$

  - Means there is an x who is a human and is a male

FOL quantifiers

$$\exists x\ \mathbf{human(x)} \land \mathbf{male(x)}$$

- It's a big OR: Equivalent to the disjunction of all the instantiations of variable x:

$$(\mathbf{human(Jerry)} \land \mathbf{male(Jerry)}) \lor$$
$$(\mathbf{human(Jing)} \land \mathbf{male(Jing)}) \lor$$
$$(\mathbf{human(laptop)} \land \mathbf{male(laptop)}) \lor \ldots$$

- Common mistake is to use $\Rightarrow$ as main connective
  - "Some pig can fly"

$$\exists x\ \mathbf{pig(x)} \Rightarrow \mathbf{fly(x)} \quad \mathbf{(wrong)}$$

FOL quantifiers

$$\exists x\ \mathbf{human(x)} \wedge \mathbf{male(x)}$$

- It's a big OR: Equivalent to the disjunction of all the instantiations of variable x:

$$(\mathbf{human(Jerry)} \wedge \mathbf{male(Jerry)}) \vee$$
$$(\mathbf{human(Jing)} \wedge \mathbf{male(Jing)}) \vee$$
$$(\mathbf{human(laptop)} \wedge \mathbf{male(laptop)}) \vee \ldots$$

- Common mistake is to use $\Rightarrow$ as main connective
  - "Some pig can fly"

$$\exists x\ \mathbf{pig(x)} \Rightarrow \mathbf{fly(x)}\ \textbf{(wrong)}$$

- This is true if there is something not a pig!

$$(\mathbf{pig(Jerry)} \Rightarrow \mathbf{fly(Jerry)}) \vee$$
$$(\mathbf{pig(laptop)} \Rightarrow \mathbf{fly(laptop)}) \vee \ldots$$

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x** $\forall$**y** is the same as $\forall$**y** $\forall$**x**
  - $\exists$**x** $\exists$**y** is the same as $\exists$**y** $\exists$**x**

- Example:
  - $\forall$**x** $\forall$**y** **`likes(x,y)`**

    Everyone likes everyone.
  - $\forall$**y** $\forall$**x** **`likes(x,y)`**

    Everyone is liked by everyone.

FOL quantifiers

- Properties of quantifiers:
  - $\forall \mathbf{x}\ \exists \mathbf{y}$ is not the same as $\exists \mathbf{y}\ \forall \mathbf{x}$
  - $\exists \mathbf{x}\ \forall \mathbf{y}$ is not the same as $\forall \mathbf{y}\ \exists \mathbf{x}$

- Example:
  - $\forall$**x** $\exists$**y** **likes(x,y)**

    Everyone likes someone (can be different).
  - $\exists$**y** $\forall$**x** **likes(x,y)**

    There is someone who is liked by everyone.

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x P(x)** when negated becomes $\exists$**x ¬P(x)**
  - $\exists$**x P(x)** when negated becomes $\forall$**x ¬P(x)**

- Example:
  - $\forall$**x sleep(x)**

    Everybody sleeps.
  - $\exists$**x ¬sleep(x)**

    Somebody does not sleep.

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x P(x)** is the same as $\neg\exists$**x** $\neg$**P(x)**
  - $\exists$**x P(x)** is the same as $\neg\forall$**x** $\neg$**P(x)**

- Example:
  - $\forall$**x sleep(x)**

    Everybody sleeps.
  - $\neg\exists$**x** $\neg$**sleep(x)**

    There does not exist someone who does not sleep.

FOL syntax

- A free variable is a variable that is not bound by an quantifier, e.g. **∃y Likes(x,y)**: **x** is free, **y** is bound
- A well-formed formula (wff) is a sentence in which all variables are quantified (no free variable)

- Short summary so far:
  - **Constants:** Bob, 2, Madison, …
  - **Variables:** *x, y, a, b, c*, …
  - **Functions:** Income, Address, Sqrt, …
  - **Predicates:** Teacher, Sisters, Even, Prime…
  - **Connectives:** $\wedge \vee \neg \Rightarrow \Leftrightarrow$
  - **Equality:** $=$
  - **Quantifiers:** $\forall \exists$

Summary

- **Term**: constant, variable, function. Denotes an object. (A ground term has no variables)
- **Atom**: the smallest expression assigned a truth value. Predicate and =
- **Sentence**: an atom, sentence with connectives, sentence with quantifiers. Assigned a truth value
- **Well-formed formula** (wff): a sentence in which all variables are quantified

Thinking in logical sentences

Convert the following sentences into FOL:

- “Elmo is a monster.”
  - What is the constant? Elmo
  - What is the predicate? Is a monster
  - Answer: monster(Elmo)

- “Tinky Winky and Dipsy are teletubbies”

- “Tom, Jerry or Mickey is not a mouse.”

Thinking in logical sentences

We can also do this with relations:

- "America bought Alaska from Russia."
  - What are the constants?
    - America, Alaska, Russia
  - What are the relations?
    - Bought
  - Answer: bought(America, Alaska, Russia)

- "Warm is between cold and hot."

- "Jerry and Jing are married."

Thinking in logical sentences

Now let's think about quantifiers:

- "Jerry likes everything."
  - What's the constant?
    - Jerry
  - Thing?
    - Just use a variable x
  - Everything?
    - Universal quantifier
  - Answer: **`∀x likes(Jerry, x)`**
  - *i.e.* **`likes(Jerry, IceCream) ∧ likes(Jerry, Jing) ∧ likes(Jerry, Armadillos) ∧ …`**
- "Jerry likes something."
- "Somebody likes Jerry."

Thinking in logical sentences

We can also have multiple quantifiers:

- “somebody heard something.”
  - What are the variables?
    - Somebody, something
  - How are they quantified?
    - Both are existential
  - Answer: `∃x,y heard(x,y)`

- “Everybody heard everything.”

- “Somebody did not hear everything.”

Thinking in logical sentences

Let's allow more complex quantified relations:

- "All stinky shoes are allowed."
  - How are ideas connected?
    - Being a shoe and being stinky implies it's allowed
  - Answer: $\forall x\ \mathbf{shoe(x)} \land \mathbf{stinky(x)} \Rightarrow \mathbf{allowed(x)}$
- "No stinky shoes are allowed."
  - Answers:
  - $\forall x\ \mathbf{shoe(x)} \land \mathbf{stinky(x)} \Rightarrow \neg \mathbf{allowed(x)}$
  - $\neg \exists x\ \mathbf{shoe(x)} \land \mathbf{stinky(x)} \land \mathbf{allowed(x)}$
  - $\neg \exists x\ \mathbf{shoe(x)} \land \mathbf{stinky(x)} \Rightarrow \mathbf{allowed(x)}$ (?)

Thinking in logical sentences

- "No stinky shoes are allowed."
  - $\neg\exists x$ shoe(x) $\wedge$ stinky(x) $\Rightarrow$ allowed(x) (?)
  - $\neg\exists x$ $\neg$(shoe(x) $\wedge$ stinky(x)) $\vee$ allowed(x)
  - $\forall x$ $\neg$ ($\neg$(shoe(x) $\wedge$ stinky(x)) $\vee$ allowed(x))
  - $\forall x$ (shoe(x) $\wedge$ stinky(x)) $\wedge$ $\neg$allowed(x)
- But this says "Jerry is a stinky shoe and Jerry is not allowed."
- How about

  $\forall x$ allowed(x) $\Rightarrow$ $\neg$ (shoe(x) $\wedge$ stinky(x))

Thinking in logical sentences

And some more complex relations:

- "No one sees everything."
- Answer: $\neg\exists x\ \forall y\ sees(x,y)$

- Equivalently: "Everyone doesn't see something."
- Answer: $\forall x\ \exists y\ \neg sees(x,y)$

- "Everyone sees nothing."
- Answer: $\forall x\ \neg\exists y\ sees(x,y)$

Thinking in logical sentences

And some *really* complex relations:

- "Any good amateur can beat some professional."
  - Ingredients: x, amateur(x), good(x), y, professional(y), beat(x,y)
  - Answer:

$$\forall x\ [\{amateur(x) \wedge good(x)\} \Rightarrow \exists y\ \{professional(y) \wedge beat(x,y)\}]$$

- "Some professionals can beat all amateurs."
  - Answer:

$$\exists x\ [professional(x) \wedge \forall y\ \{amateur(y) \Rightarrow beat(x,y)\}]$$

Thinking in logical sentences

We can throw in functions and equalities, too:

- "Jerry and Jing are the same age."
  - Are functional relations specified?
  - Are equalities specified?
  - Answer: `age(Jerry) = age(Jing)`
- "There are exactly two shoes."
  - ?

Thinking in logical sentences

- "There are exactly two shoes."
  - First try:

$\exists x\ \exists y\ shoe(x) \land shoe(y)$

Thinking in logical sentences

- "There are exactly two shoes."
  - First try:

$\exists x\ \exists y\ shoe(x) \land shoe(y)$

  - Second try:

$\exists x\ \exists y\ shoe(x) \land shoe(y) \land \neg(x=y)$

Thinking in logical sentences

- "There are exactly two shoes."
  - First try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y)$

  - Second try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y) \wedge \neg(x=y)$

  - Third try:

$$\exists x\ \exists y\ shoe(x) \wedge shoe(y) \wedge \neg(x=y) \wedge \forall z\ (shoe(z) \Rightarrow (x=z) \vee (y=z))$$

Thinking in logical sentences

- Interesting words: **always, sometimes, never**
  - “Good people *always* have friends.”

Thinking in logical sentences

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  - "Good people *always* have friends."

    $\forall$**`x person(x)`** $\wedge$ **`good(x)`** $\Rightarrow$ $\exists$**`y(friend(x,y))`**
  - "Busy people *sometimes* have friends."

Thinking in logical sentences

- Interesting words: **always, sometimes, never**
  - "Good people *always* have friends."
    $\forall x\ person(x) \wedge good(x) \Rightarrow \exists y(friend(x,y))$
  - "Busy people *sometimes* have friends."
    $\exists x\ person(x) \wedge busy(x) \wedge \exists y(friend(x,y))$
  - "Bad people *never* have friends."

Thinking in logical sentences

- Interesting words: **always, sometimes, never**
  - "Good people *always* have friends."

    $\forall x\ \mathbf{person(x)} \wedge \mathbf{good(x)} \Rightarrow \exists y(\mathbf{friend(x,y)})$
  - "Busy people *sometimes* have friends."

    $\exists x\ \mathbf{person(x)} \wedge \mathbf{busy(x)} \wedge \exists y(\mathbf{friend(x,y)})$
  - "Bad people *never* have friends."

    $\forall x\ \mathbf{person(x)} \wedge \mathbf{bad(x)} \Rightarrow \neg\exists y(\mathbf{friend(x,y)})$

Thinking in logical sentences

Tricky sentences

- "x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y."

Thinking in logical sentences

Tricky sentences

- "x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y."

$$\forall x\ \forall y\ \mathbf{above}(x,y) \Leftrightarrow$$

$$[\mathbf{onTop}(x,y) \lor \exists z\{\mathbf{onTop}(x,z) \land \mathbf{above}(z,y)\}]$$

Next: Inference for FOL

- Recall that in propositional logic, inference is easy
  - Enumerate all possibilities (truth tables)
  - Apply sound inference rules on facts

- But in FOL, we have the concepts of variables, relations, and quantification
  - This complicates things quite a bit!

- We will discuss inference in FOL next time.