1. For the following choice of $A$ and $b$ solve the system of equations $Ax = b$. If there are multiple solutions, describe the full solution set. If there are linear dependence relations between the rows of the coefficient matrix, state them.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

**Solution:**

First form a tableau

$$
\begin{array}{cccc|c}
  & x_1 & x_2 & x_3 & 1 \\
y_1 & 1 & -2 & -1 & -2 \\
y_2 & -1 & -1 & 0 & 1 \\
\end{array}
$$

Swap $x_3$ and $y_1$:

$$
\begin{array}{cccc|c}
  & x_1 & x_2 & y_1 & 1 \\
x_3 & 1 & -2 & -1 & -2 \\
y_2 & -1 & -1 & 0 & 1 \\
\end{array}
$$

Then swap $x_1$ and $y_2$:

$$
\begin{array}{cccc|c}
  & y_2 & x_2 & y_1 & 1 \\
x_3 & -1 & -3 & -1 & -1 \\
x_1 & -1 & -1 & 0 & 1 \\
\end{array}
$$

To read the tableau, we set $y_1$ and $y_2$ equal to zero. If we set $x_2 = t$, then we find the solution set equals

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} : t \in \mathbb{R} \right\}.$$
2. Consider the following linear program

\[
\begin{align*}
\text{minimize} & \quad 9x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq 4 \\
& \quad 3x_1 - x_2 \geq -2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

(a) Write down the dual of this problem.

(b) Find solutions for the primal and dual.

(c) Suppose the right-hand side of the first constraint is changed from 4 to 6. Without performing any additional simplex iterations or referring to the tableau, give a lower bound on the optimal primal objective value of the modified problem. Explain.

**Solution:**

(a) The dual problem is

\[
\begin{align*}
\text{maximize} & \quad 4u_1 - 2u_2 \\
\text{subject to} & \quad u_1 + 3u_2 \leq 9 \\
& \quad u_1 - u_2 \leq 1 \\
& \quad u_1, u_2 \geq 0
\end{align*}
\]

(b) Observe that \((0,0)\) is feasible for the dual. So we will solve this by the dual simplex method. Form the Tableau:

\[
\begin{array}{cccc|c}
\hline
u_3 & u_4 & x_1 & x_2 & 1 \\
-1u_1 & x_3 & 1 & 1 & -4 \\
-1u_2 & x_4 & 3 & -1 & 2 \\
\hline
9 & 1 & 0
\end{array}
\]

Swap \(u_1\) and \(u_4\)

\[
\begin{array}{cccc|c}
\hline
u_3 & u_1 & x_1 & x_3 & 1 \\
-1u_4 & x_2 & -1 & 1 & 4 \\
-1u_2 & x_4 & 4 & -1 & -2 \\
\hline
8 & 1 & 4
\end{array}
\]

And then swap \(u_2\) and \(u_3\)

\[
\begin{array}{cccc|c}
\hline
u_2 & u_1 & x_4 & x_3 & 1 \\
-1u_4 & x_2 & -1/4 & 3/4 & 7/2 \\
-1u_3 & x_1 & 1/4 & 1/4 & 1/2 \\
\hline
2 & 3 & 8
\end{array}
\]
As there are no pivots to improve the cost, we terminate at an optimal solution:

\[(\bar{x}_1, \bar{x}_2) = (1/2, 3/2), \quad (\bar{u}_1, \bar{u}_2) = (3, 2)\]

(c) If we change the right hand side from 4 to 6, this changes the dual objective but not the dual constraints. Thus, our dual optimal solution \((3, 2)\) is feasible for the modified dual, and, by weak duality, provides us with the lower bound \(6 \times 3 - 2 \times 2 = 14\) of the modified primal objective.
3. Consider the following linear program, where \( c_1, c_2, c_3 \) are constants:

\[
\text{maximize } \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to } \quad -1 \leq x_1 \leq 1 \\
\quad \quad \quad \quad -1 \leq x_2 \leq 1 \\
\quad \quad \quad \quad -1 \leq x_3 \leq 1
\]

(a) Write down the dual of this problem.
(b) Write down the KKT conditions for this problem.
(c) Find optimal solutions of the primal and dual problems that jointly satisfy the KKT conditions.
(d) Write the optimal cost of the primal problem solely in terms of the constants \( c_1, c_2, c_3 \).

Solution:

(a) Write the primal out in standard dual form:

\[
\text{maximize } \quad c_1 x_1 + c_2 x_2 + c_3 x_3 \\
\text{subject to } \quad x_1 \leq 1 \\
\quad \quad \quad \quad -x_1 \leq 1 \\
\quad \quad \quad \quad x_2 \leq 1 \\
\quad \quad \quad \quad -x_2 \leq 1 \\
\quad \quad \quad \quad x_3 \leq 1 \\
\quad \quad \quad \quad -x_3 \leq 1
\]

Then the dual is

\[
\text{minimize } \quad u_1 + v_1 + u_2 + v_2 + u_3 + v_3 \\
\text{subject to } \quad u_1 - v_1 = c_1 \\
\quad \quad \quad \quad u_2 - v_2 = c_2 \\
\quad \quad \quad \quad u_3 - v_3 = c_3 \\
\quad \quad \quad \quad u_1, u_2, u_3 \geq 0 \\
\quad \quad \quad \quad v_1, v_2, v_3 \geq 0
\]

(b) The KKT conditions are

\[
0 \leq 1 - x_1 \perp u_1 \geq 0 \\
0 \leq 1 + x_1 \perp v_1 \geq 0 \\
0 \leq 1 - x_2 \perp u_2 \geq 0 \\
0 \leq 1 + x_2 \perp v_2 \geq 0 \\
0 \leq 1 - x_3 \perp u_3 \geq 0 \\
0 \leq 1 + x_3 \perp v_3 \geq 0 \\
u_1 - v_1 = c_1 \\
u_2 - v_2 = c_2 \\
u_3 - v_3 = c_3
\]
(c) If $c_i > 0$, set $u_i = c_i$, $v_i = 0$, and $x_i = 1$. If $c_i < 0$, set $u_i = 0$, $v_i = -c_i$, and $x_i = -1$. If $c_i = 0$, set $u_i = 0$, $v_i = 0$, and $x_i = 0$. It is readily verified that in all cases, the KKT conditions are satisfied.

(d) Using the dual optimal assignments, we see that $u_i + v_i = |c_i|$. Therefore, the optimal dual cost is equal to $|c_1| + |c_2| + |c_3|$. By strong duality, this is also equal to the optimal primal cost.