CS726 - Nonlinear Optimization I - Homework I

September 19, 2012

This assignment is due at the beginning of class on September 26.

1. The α-sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set $S_\alpha = \{x \ : \ f(x) \leq \alpha\}$.

(a) Suppose f is convex. Show that S_α is convex for all $\alpha \in \mathbb{R}$.

(b) Suppose S_α is convex for all $\alpha \in \mathbb{R}$. Is f convex? Prove or find a counterexample.

(c) Suppose f is convex. Show that the set of global minimizers of f is a convex set.

2. A function $q : \mathbb{R}^n \to \mathbb{R}^n$ is monotone if

$$(q(x) - q(y))^T (x - y) \geq 0$$

for all x and y.

(a) Let f be a differentiable convex function. Show $\nabla f(x)$ is monotone.

(b) Let

$$q(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

Show that q is montone on $\mathbb{R}$.

(c) Is every monotone mapping the gradient of a convex function? Explain your reasoning.

3. Let the *support function* of a set C be defined as

$$S_C(x) = \sup_{y \in C} x^T y$$

(a) Show S_C is convex.

(b) Show that $S_{A+B} = S_A + S_B$.

(c) Show $S_{A \cup B} = \max\{S_A, S_B\}$.

(d) Let B be closed and convex. Show that $A \subseteq B$ if and only if $S_A(y) \leq S_B(y)$ for all y.

4. Consider the function

$$f(x, y) = \tfrac{1}{2}x^2 + \tfrac{1}{4}y^4 - \tfrac{1}{2}y^2$$

(a) Compute the gradient and Hessian of f.

(b) Compute the set of points where $\nabla f = 0$. For each point, determine if it is a local minima, local maxima, or a global minimum.

(c) Consider the gradient method with exact line search starting at the point $(x_0, y_0) = (1, 0)$. Determine to which point in $\{x \ : \ \nabla f(x) = 0\}$ the algorithm converges. Explain your reasoning.