CS726 - Nonlinear Optimization I - Homework 4

October 3, 2012

This assignment is due at the beginning of class on October 10.
In all of these problems, you can check your results on a computer, but your solution to this problem should be analytic.

1. Consider the function $f(x) = \frac{1}{2}x_1^2 + \frac{\gamma}{2}x_2^2$ where $0 < \gamma < 1$.

(a) What is the largest condition number of the Hessian of $f(x)$ for $x \in \mathbb{R}^2$?

(b) Perform gradient descent with exact line search starting at the point $x^{(0)} = (\gamma, 1)$. Determine a closed form expression for the kth iterate, $x^{(k)}$.

(c) Determine a tight bound for $f(x^{(k)})$ in terms of γ, k, and $f(x^{(0)})$. What is the convergence rate?

2. Consider the functions $f_k : \mathbb{R}^n \to \mathbb{R}$, $f_k(x) = \frac{1}{2}\left(x_1^2 + \sum_{i=1}^{k-1}(x_i - x_{i+1})^2 + x_k^2\right) - x_1$.

(a) Compute the set of minimizers of f_k for $k = 1, \ldots, n$. Also compute the function value at these minimizers.

(b) Suppose we wish to minimize $f = f_n$ by the gradient method with $x^{(0)} = 0$. Show that after k iterations $x_j^{(k)}$ is equal to zero if $j > k$.

(c) Show that $f_n(x^{(k)}) = f_k(x^{(k)})$.

(d) Show that $f(x^{(k)}) - f_*$ does not converge linearly to 0.

3. Let f be a smooth function.

(a) Show that if f is strongly convex, then the Netwon direction is a descent direction.

(b) Find a convex function for which the Newton direction does not exist for some non-minimizing x.

4. Apply Newton's method with a constant stepsize to minimize the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) := \frac{1}{3}\left(\sum_{i=1}^{n} x_i^2\right)^{3/2}.$$

Identify the range of stepsizes for which this method converges. Show that for any stepsize within this range, the iterates converge linearly to $x_{\text{opt}} = 0$. Explain why the method does not converge quadratically to the optimal solution.