CS726 - Nonlinear Optimization I - Homework 7

October 31, 2012

This assignment is due at the beginning of class on November 7.

1. Consider the linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

Here, A is $m \times n$. Let $p^\star$ denote the optimal value.

(a) Let $w \in \mathbb{R}^m$ have nonnegative components. Show that the feasible set is contained in the half space

$$H_w = \{x \;:\; w^T A x \le w^T b\}\,.$$

(b) Let $p_w = \inf\{c^T x \;:\; x \in H_w\}$. Show that $p_w \le p^\star$.

(c) Derive an expression for p_w.

(d) Relate the problem of maximizing p_w with respect to $w \ge 0$ to the dual of the original linear program.

2. Consider the equality constrained problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax = b \end{array}$$

where f_0 is convex and differentiable and $A \in \mathbb{R}^{m\times n}$ has rank m. The quadratic penalty method solves the unconstrained problem

$$\text{minimize} \quad \phi(x) = f_0(x) + \alpha\|Ax - b\|^2$$

where $\alpha > 0$ is a penalty parameter. The idea is that a minimizer of the auxiliary function, $\tilde{x}$, should be an approximate solution of the original problem. Intuition suggests that the larger the penalty weight α, the better the approximation $\tilde{x}$ will be to a solution of the original problem. Suppose $\tilde{x}$ is a minimizer of ϕ. Show how to find, from $\tilde{x}$, a dual feasible point for the original equality constrained problem. Find the corresponding lower bound on the optimal value of the original problem.

3. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & Gx = h \end{array}$$

Where A is $m \times n$, $m \ge n$, and the rank of A is n. G is $p \times n$, $p \le n$, and the rank of G is p. Write down the KKT conditions and derive expressions for the optimal primal and dual solutions.

4. Given two, finite sets of points $X = \{x_1, \ldots, x_p\}$ and $Z = \{z_1, \ldots, z_m\}$, suppose we wish to find a hyperplane such that X lies on one side of the hyperplane and Z lies on the other. One way to find such a hyperplane would be to solve the quadratic program

$$\begin{array}{ll} \text{minimize}_(a, b) & \|a\|_2^2 \\ \text{subject to} & a^T x_k + b \geq 1 \ \text{ for } \ k = 1, \ldots, p \\ & a^T z_j + b \leq -1 \ \text{ for } \ j = 1, \ldots, m \end{array}$$

Show that the optimal $a^\star$ has the form

$$a^\star = \sum_{k=1}^{p} \beta_k x_k - \sum_{j=1}^{m} \gamma_j z_j$$

where $\beta \geq 0$, $\gamma \geq 0$, and $\sum_{k=1}^{p} \beta_k = \sum_{j=1}^{m} \gamma_j$.