Bregman Iterative method for l1-minimization

Author: W. Yin, S. Osher, D. Goldfarb, J. Darbon
Presenter: Shih-Hsuan Hsu
ECE Department

Outline

- Bregman Distance
- Generalized Bregman Iteration
- Bregman Iteration for Basis Pursuit
- Properties of Bregman Iteration
- Linearized Bregman Iteration (If time permits)

Bregman Distance

The *Bregman distance* of J between u and v is

$$D_J^p(u,v) = J(u) - J(v) - \langle p, u-v \rangle, \quad p \in \partial J(v).$$

$$\partial J(v) := \{p : J(u) \geq J(v) + \langle p, u-v \rangle, \forall u\}$$

. Bregman distance satisfies

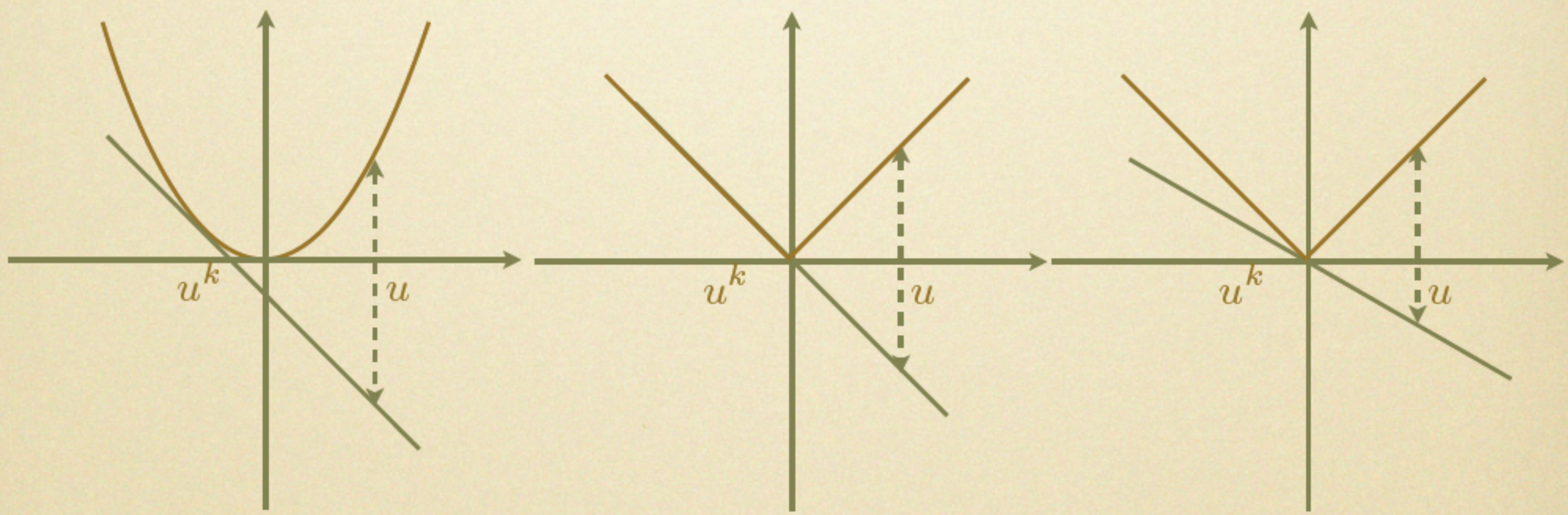
(a) $D_J^p(v,v) = 0$

(b) $D_J^p(u,v) \geq 0$

(c) $D_J^p(u,v) + D_J^{\tilde{p}}(v,\tilde{v}) - D_J^{\tilde{p}}(u,\tilde{v}) = \langle p - \tilde{p}, v - u \rangle$

Bregman distance

$$D(u, u^k) := J(u) - J(u^k) - < \partial J(u^k), u - u^k >$$



Bregman Distance

Generally, the Bregman Distance for

(u,v) and (v,u) is in general not equal, it is not a distance in usual sense.

Measure distance between u and v and satisfies that :

if w lies on the line segment of (u,v) ,

then (u,w) has smaller Bregman distance than (u,v) does

Generalized Bregman Iteration

- Given functions J() and H(), to solve:

$$\min_u J(u) + H(u)$$

Initialize: $k = 0$, $u^0 = \mathbf{0}$, $p^0 = \mathbf{0}$.
while "not converge" **do**
$\quad u^{k+1} \leftarrow \arg\min_u D_J^{p^k}(u, u^k) + H(u)$
$\quad p^{k+1} \leftarrow p^k - \nabla H(u^{k+1}) \in \partial J(u^{k+1})$
$\quad k \leftarrow k + 1$
end while

Generalized Bregman Iteration

- $\{H(u_k)\}$ decreases monotonically

Proof. Since u_{k+1} minimizes $D_J^{p_k}(u, u_k) + \lambda H(u)$,

$$\begin{aligned} \lambda H(u_{k+1}) &\leq D_J^{p_k}(u_{k+1}, u_k) + \lambda H(u_{k+1}) \\ &\leq D_J^{p_k}(u_k, u_k) + \lambda H(u_k) \\ &= \lambda H(u_k). \end{aligned}$$

Generalized Bregman Iteration

Suppose H is differentiable let $u^\star$ be a minimizer of H, then

$$H(u_k) \le H(u^\star) + \frac{D_J^{p_0}(u^\star, u_0)}{\lambda k}.$$

Proof.

$$\begin{aligned}
D_J^{p_k}(u^\star, u_k) - D_J^{p_{k-1}}(u^\star, u_{k-1}) &\le D_J^{p_k}(u^\star, u_k) + D_J^{p_{k-1}}(u_k, u_{k-1}) - D_J^{p_{k-1}}(u^\star, u_{k-1}) \\
&= \langle p_k - p_{k-1}, u_k - u^\star \rangle \\
&= \langle \lambda \nabla H(u_k), u^\star - u_k \rangle \\
&\le \lambda\big(H(u^\star) - H(u_k)\big), \\
&\le \lambda\big(H(u^\star) - H(u_K)\big)
\end{aligned}$$

Summing over $k = 1, \ldots K$

$$\begin{aligned}
\sum_{k=1}^{K} D_J^{p_k}(u^\star, u_k) - D_J^{p_{k-1}}(u^\star, u_{k-1}) + \lambda\big(H(u_k) - H(u^\star)\big) &\le 0 \\
D_J^{p_K}(u^\star, u_K) - D_J^{p_0}(u^\star, u_0) + \lambda K\big(H(u_K) - H(u^\star)\big) &\le 0 \\
-D_J^{p_0}(u^\star, u_0) + \lambda K\big(H(u_K) - H(u^\star)\big) &\le 0.
\end{aligned}$$

Bregman Iteration for Basis Pursuit

- Basis Pursuit: Constrained Optimization Problem

$$\min_u \{\|u\|_1 : Au = f\}$$

Convert to Unconstrained formulation:

$$\min_u \mu\|u\|_1 + \frac{1}{2}\|Au - f\|_2^2$$

Now let:

$$J(u) = \mu\|u\|_1 \text{ and } H(u) = \tfrac{1}{2}\|Au - f\|^2:$$

Bregman Iteration for Basis Pursuit

Version 1:

$$u^0 \leftarrow \mathbf{0},\ p^0 \leftarrow \mathbf{0},$$
$$\text{For } k = 0, 1, \ldots \text{ do}$$
$$u^{k+1} \leftarrow \arg\min_u D_J^{p^k}(u, u^k) + \frac{1}{2}\|Au - f\|^2,$$
$$p^{k+1} \leftarrow p^k - A^\top (Au^{k+1} - f);$$

Version 2:

$$f^0 \leftarrow \mathbf{0},\ u^0 \leftarrow \mathbf{0},$$
$$\text{For } k = 0, 1, \ldots \text{ do}$$
$$f^{k+1} \leftarrow f + (f^k - Au^k),$$
$$u^{k+1} \leftarrow \arg\min_u J(u) + \frac{1}{2}\left\|Au - f^{k+1}\right\|^2.$$

Properties

- It can be shown that:

$$A^\top(Au^{k+1} - f) = A^\top(A\bar{u}^{k+1} - f)$$

- Base on this we can show that p^{k+1} is actually:

$$\begin{aligned} p^{k+1} = p^k - A^\top(Au^{k+1} - f) &= p^k - A^\top(A\bar{u}^{k+1} - f) \\ &= A^\top(f^k - A\bar{u}^k) - A^\top(A\bar{u}^{k+1} - f) \\ &= A^\top\left(f + (f^k - A\bar{u}^k) - A\bar{u}^{k+1}\right) \\ &= A^\top(f^{k+1} - A\bar{u}^{k+1}). \end{aligned}$$

Properties:

- Objective function in two version's subproblem are the same (up to a constant of u)

$$
\begin{aligned}
D_J^{p^k}(u, u^k) + \frac{1}{2}\|Au - f\|^2 &= J(u) - \langle p^k, u\rangle + \frac{1}{2}\|Au - f\|^2 + C_1 \\
&= J(u) - \langle f^k - A\bar{u}^k, Au\rangle + \frac{1}{2}\|Au - f\|^2 + C_2 \\
&= J(u) + \frac{1}{2}\|Au - (f + (f^k - A\bar{u}^k))\|^2 + C_3 \\
&= J(u) + \frac{1}{2}\|Au - f^{k+1}\|^2 + C_3,
\end{aligned}
$$

Once feasible, it's Optimal:

- Suppose the iterate u^k in version 1 satisfies: $Au^k = f$, then u^k is a solution of the basis pursuit problem

Proof.

For any u, by the nonnegativity of the Bregman distance, we have

$$\begin{aligned} J(u^k) &\leq J(u) - \langle u - u^k, p^k \rangle \\ &= J(u) - \langle u - u^k, A^\top (f^k - Au^k) \rangle \\ &= J(u) - \langle Au - Au^k, f^k - Au^k \rangle \\ &= J(u) - \langle Au - f, f^k - f \rangle, \\ &= J(u) \qquad \text{for any } u \text{ satisfying } Au = f \end{aligned}$$

Convergence Bregman Iteration

- There exist a number K such that for any k≥K u^k is a solution to the basis pursuit problem

  let $\tilde{u}$ satisfy $H(\tilde{u}) = \frac{1}{2}\|A\tilde{u} - f\|^2 = 0.$

  From property of generalized Bregman Iteration:

$$H(u_k) \leq H(\tilde{u}) + \frac{D_J^{p_0}(\tilde{u}, u_0)}{\lambda k} = \frac{D_J^{p_0}(\tilde{u}, u_0)}{\lambda k}.$$

Convergence of Bregman Iteration

Let (I_+^j, I_-^j, E^j) be a partition of the index set $\{1, 2, \ldots, n\}$, and define

$$U^j := U(I_+^j, I_-^j, E^j) = \{u : u_i \geq 0,\ i \in I_+^j;\ u_i \leq 0,\ i \in I_-^j;\ u_i = 0,\ i \in E^j\},$$

$$H^j := \min_u \left\{ \frac{1}{2} \|Au - f\|^2 : u \in U^j \right\}.$$

for each j with $H^j > 0$ there is a sufficiently large K_j such that

u^k is not in U^j for $k \geq K_j$

$K := \max_j \{K_j : H^j > 0\}$, we have $H(u^k) = 0$ for $k \geq K$.

$$Au^k = f \text{ for } k \geq K$$

Bregman Iteration for Basis Pursuit

- Why is u^k and p^k this way? (Consider Version 1)

  u^{k+1} remain close to u^k and the penalty term keep the updates not too far from the feasible set {u | Au=f }

  p^{k+1} : for version 1, u^{k+1} satisfies the first-order optimality condition, so

$$\mathbf{0} \in \partial J(u^{k+1}) - p^k + \nabla H(u^{k+1}) = \partial J(u^{k+1}) - p^k + A^\top(Au^{k+1} - f).$$

$$p^{k+1} = p^k - A^\top(Au^{k+1} - f) \in \partial J(u^{k+1});$$

Linearized Bregman Iteration

- H() is linearly approximated with a quadratic penalty term:

$$H(u) \approx H(u_k) + \nabla H(u_k) \cdot (u - u_k).$$

$$\arg\min_u \; J(u) + H(u)$$

$$u^{k+1} \leftarrow \arg\min_u \; J(u) + H(u_k) + \langle \nabla H(u_k), u - u_k \rangle$$

$$u^{k+1} \leftarrow \arg\min_u \; J(u) + \langle \nabla H(u_k), u - u_k \rangle \quad \frac{1}{2\delta} \|u - u_k\|_2^2.$$

$$u^{k+1} \leftarrow \arg\min_u \; J(u) + \frac{1}{2\delta} \left\| u - \left(u^k - \delta \nabla H(u_k) \right) \right\|^2$$

$$u^{k+1} \leftarrow \arg\min_u \; J(u) + \frac{1}{2\delta} \left\| u - \left(u^k - \delta A^\top (A u^k - f) \right) \right\|^2$$

For Basis Pursuit problem

- From optimality condition of subproblem:

$$\mathbf{0} = p^{k+1} - p^k + \frac{1}{\delta}\left(u^{k+1} - \left(u^k - \delta A^\top (Au^k - f)\right)\right),$$

- Since $p^0 = \mathbf{0}$ and $u^0 = \mathbf{0}$,

$$p^{k+1} = p^k - A^\top (Au^k - f) - \frac{(u^{k+1} - u^k)}{\delta}$$

$$= \cdots = \sum_{j=0}^{k} A^\top (f - Au^j) - \frac{u^{k+1}}{\delta}.$$

Linearized Bregman Iteration

- Let

$$v^k = \sum_{j=0}^{k} A^\top (f - Au^j),$$

$$\begin{aligned} u_i^{k+1} &\leftarrow \delta \, \text{shrink}(v_i^k, \mu), \quad i = 1, \ldots, n, \\ v^{k+1} &\leftarrow v^k + A^\top (f - Au^{k+1}). \end{aligned}$$