Nuclear norm minimization for the planted clique and biclique problems

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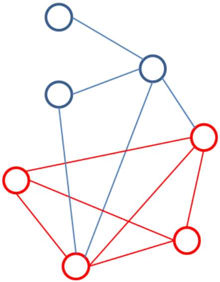
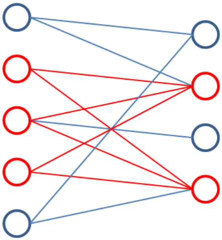
Problems of interest

- Maximum clique problem :
  Given an undirected graph (V,E), find the largest clique.
- Maximum-edge biclique problem :
  Given a bipartite graph (U,V,E), find the complete bipartite subgraph $K_{m,n}$ that maxmizes the product mn.





Results on norms of random matrices (1/3)

Consider a random matrix A with i.i.d. entries following distribution Ω. Define Ω as follows:

$$A_{ij} = \begin{cases} 1 & \text{with probability p} \\ -\frac{p}{(1-p)} & \text{with probability 1-p} \end{cases}$$

Then mean$(A_{ij}) = 0$, variance$(A_{ij}) = \sigma^2 = \frac{p}{1-p}$

Theorem

For all integers i,j, $1 \leq j \leq i \leq n$, let A_{ij} be distributed according to Ω, and define define symmetrically $A_{ij} = A_{ji}$ for all $i < j$. Then the random symmetric matrix $A = [A_{ij}]$ satisfies

$$\|A\| \leq 3\sigma\sqrt{n}$$

w.p. at least $1 - exp(-cn^{1/6})$ for some constant $c > 0$ depending on σ.

Results on norms of random matrices (2/3)

Theorem

Let A be a $\lceil yn \rceil \times n$ matrix whose entries are chosen according to Ω for fixed $y \in \mathbb{R}_+$. Then, w.p. at least $1 - c_1 exp(-c_2 n^{c_3})$ where $c_1, c_2, c_3 > 0$ depend on p and y,

$$\|A\| \leq c_4 \sqrt{n}$$

for some $c_4 > 0$ also depending on p, y.

Theorem (Chernoff Bounds)

Let $X_1, \cdots, X_k$ be a sequence of k independent Bernoulli trials with success probability p, and let $S = \sum_{i=1}^{k} X_i$. Then

$$P(S > (1+\delta)pk) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{pk} \text{ for } \delta > 0, \text{ and}$$

$$P(|S - pk| > a\sqrt{k}) \leq 2exp(-a^2/p) \quad \forall a \in (0, p\sqrt{k}).$$

Results on norms of random matrices (3/3)

Theorem

Let A be an $n \times N$ matrix whose entries are chosen according to Ω. Let $\tilde{A}$ be defined as follows. For (i,j) such that $A_{ij} = 1$, define $\tilde{A}_{ij} = 1$. For (i,j) such that $A_{ij} = -p/(1-p)$, take $\tilde{A}_{ij} = -n_j/(n - n_j)$, where n_j is the number of 1's in column j of A. Then there exist $c_1 > 0$ and $c_2 \in (0, 1)$ depending on p such that

$$P(\|A - \tilde{A}\|_F^2 \leq c_1 N) \geq 1 - (2/3)^N - N c_2^n.$$

Maximum clique: Formulation (1/2)

Let G=(V,E) be a simple graph. For any clique K of G, the adjacency matrix of the graph K' obtained by taking the union of K and the set of loops for each $v \in V(K)$ is a rank-one matrix with 1's in the entries indexed by $V(K) \times V(K)$, and 0's everywhere else. Therefore, a clique K of G containing n vertices can be found by solving the following rank minimization problem.

$$\begin{aligned} &\min \; rank(X) \\ &s.t. \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, \\ &\qquad X_{ij} = 0 \quad \text{if } (i,j) \notin E \text{ and } i \neq j, \\ &\qquad X \in [0,1]^{V \times V}. \end{aligned}$$

Maximum clique: Formulation (2/2)

Underestimating $rank(X)$ with $\|X\|_*$, we obtain the following convex optimization problem.

$$
\begin{aligned}
(P_0) \quad & \min \|X\|_* \\
& s.t. \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, \\
& \quad X_{ij} = 0 \quad \text{if } (i,j) \notin E \text{ and } i \neq j.
\end{aligned}
$$

We generalize it to

$$
\begin{aligned}
(P) \quad & \min \|X\|_* \\
& s.t. \sum_{i \in V} \sum_{j \in V} X_{ij} \geq mn, \\
& \quad X_{ij} = 0 \quad \text{if } (i,j) \in \tilde{E},
\end{aligned}
$$

where $X \in \mathbb{R}^{M \times N}$, $E \subseteq \{1, \cdots, M\} \times \{1, \cdots, N\}$, and $\tilde{E}$ is the complement of E.

Maximum clique: optimality conditions (1/2)

Lemma

Suppose $A \in \mathbb{R}^{m \times n}$ has rank r with svd $A = \sum_{k=1}^{r} \sigma_k u_k v_k^T$. Then ψ is a subgradient of $\|\cdot\|_*$ iff ψ is of the form

$$\psi = \sum_{k=1}^{r} u_k v_k^T + W,$$

where W satisfies $\|W\| \leq 1$ and the column space of W is orthogonal to u_k and the row space of W is orthogonal to v_k for all $k = 1, \cdots, r$.

Using this, we can derive the optimality condition for (P) by straightforward application of KKT conditions.

Maximum clique: optimality conditions (2/2)

Theorem

Let $U^* \subset \{1, \cdots, M\}$ with $|U^*| = m$, and $V^* \subset \{1, \cdots, N\}$ with $|V^*| = n$. Let $\bar{u}, \bar{v}$ be the characteristic vectors of U^*, V^*, respectively. Suppose $X^* = \bar{u}\bar{v}^T$ is feasible for (P) and there exists $W \in \mathbb{R}^{M \times N}$, $\lambda \in \mathbb{R}^{M \times N}$, and $\mu \in \mathbb{R}_+$ such that

$$
\begin{aligned}
W\bar{v} = 0, \bar{u}^T W = 0, \|W\| \leq 1, \ \text{and} \\
\frac{\bar{u}\bar{v}^T}{\sqrt{mn}} + W = \mu e e^T + \sum_{(i,j) \in \tilde{E}} \lambda_{ij} e_i e_j^T.
\end{aligned}
$$

Then X^* is an optimal solution to (P). Moreover, for any $I \subset \{1, \cdots, M\}$, $J \subset \{1, \cdots, N\}$ such that $I \times J \subset E$, $|I| \cdot |J| \leq mn$. Furthermore, if $\|W\| < 1$ and $\mu > 0$, then X^* is the unique optimizer.

Maximum clique: finding the solution

Given a graph G, how do we find X^*, W, and values for the multipliers to satisfy the optimality conditions?

$\rightarrow$ Take $\mu = 1/n$ and define W, λ as the following:

- If $(i,j) \in V^* \times V^*$, $W_{ij} = 0$, $\lambda_{ij} = 0$.
- If $(i,j) \in E - (V^* \times V^*)$ such that $i \neq j$, $W_{ij} = 1/n$, $\lambda_{ij} = 0$.
- If $i \notin V^*$, $W_{ii} = 1/n$.
- If $(i,j) \notin E$, $i \notin V^*$, $j \notin V^*$, then $W_{ij} = -\gamma/n$, $\lambda_{ij} = -(1+\gamma)/n$ for some constant $\gamma \in \mathbb{R}$.
- If $(i,j) \notin E$, $i \in V^*$, $j \notin V^*$,

  $$W_{ij} = -\frac{p_j}{n(n-p_j)},\ \lambda_{ij} = -\frac{1}{n} - \frac{p_j}{n(n-p_j)},$$

  where p_j is the number of edges from j to V^*.
- If $(i,j) \notin E$, $i \notin V^*$, $j \in V^*$, choose W_{ij}, λ_{ij} symmetrically with the previous case.

Maximum clique: finding the solution

We can easily check that this W satisfies

$$W\bar{v} = 0,$$
$$\frac{\bar{v}\bar{v}^T}{n} + W = \mu ee^T + \sum_{(i,j)\in\tilde{E}} \lambda_{ij} e_i e_j^T.$$

Which graphs G yield W defined as suggested such that $\|W\| < 1$?

- Adversarial case ($\gamma = 0$)
- Randomized case ($\gamma = -p/(1-p)$)

Maximum clique: the randomized case

Let V be a set of N vertices, and consider a subset $V^* \subset V$ with n vertices. We construct the edge set E as follows:

- For all $(i,j) \in V^* \times V^*$, $(i,j) \in E$
- Each of remaining $N(N-1)/2 - n(n-1)/2$ possible edges is added to E independently at random w.p. $p \in [0,1)$

We wish to determine which n, N yield G as constructed above such that with high probability $X^* = \bar{v}\bar{v}^T$ is optimal for the convex relaxation (P_0) of the clique problem.

Maximum clique: the randomized case

Theorem

There exists an $\alpha > 0$ depending on p such that for all G constructed as suggested with $n \geq \alpha\sqrt{N}$, the clique defined by $V^* \times V^*$ is the unique maximum clique of G and will correspond to the unique solution of P_0 with probability tending exponentially to 1 as $N \to \infty$.

Conclusion

Maximum clique and maximum biclique problems, both of which are NP-hard, can be solved in polynomial time using nuclear norm minimization, provided that the input graph consists of a single clique or biclique plus diversionary edges.