Vandermonde Decomposition of the Prony Hankel Matrix

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Let $S = \{n_1, ..., n_s\}$ be the indices of x corresponding to nonzero values so that $x_S := \{x_{n_1}, .., x_{n_s}\}$ is the support of x. Then, the DFT coefficients are given by

$$\hat{x}_k = \sum_{n \in S} x_n \omega_D^{nk} \tag{1}$$

$$= \sum_{l=1}^{s} h_l t_l^D \qquad \text{where } h_l := x_{n_l} \neq 0 \text{ and } t_l := \omega_D^{n_l} \tag{2}$$

so that

$$\hat{x}_{m+k} = (h_1 t_1^m) t_1^k + \cdots + (h_s t_s^m) t_s^k.$$

Then the $(m+1)$th column in the "Prony" Hankel matrix can be written as

$$\begin{pmatrix} \hat{x}_m \\ \vdots \\ \hat{x}_{m+k} \\ \vdots \\ \hat{x}_{m+s-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_s \\ \vdots & \vdots & \dots & \vdots \\ t_1^k & t_2^k & \dots & t_s^k \\ \vdots & \vdots & \dots & \vdots \\ t_1^{s-1} & t_2^{s-1} & \dots & t_s^{s-1} \end{pmatrix} \begin{pmatrix} h_1 t_1^m \\ \vdots \\ h_k t_k^m \\ \vdots \\ h_s t_s^m \end{pmatrix}. \tag{3}$$

Thus the Prony Matrix can be written as

$$\begin{pmatrix} \hat{x}_0 & \dots & \hat{x}_m & \dots & \hat{x}_{s-1} \\ \vdots & & \vdots & & \vdots \\ \hat{x}_k & \dots & \hat{x}_{m+k} & \dots & \hat{x}_{k+s-1} \\ \vdots & & \vdots & & \vdots \\ \hat{x}_{s-1} & \dots & \hat{x}_{s+k-1} & \dots & \hat{x}_{2s-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_s \\ \vdots & \vdots & \dots & \vdots \\ t_1^k & t_2^k & \dots & t_s^k \\ \vdots & \vdots & \dots & \vdots \\ t_1^{s-1} & t_2^{s-1} & \dots & t_s^{s-1} \end{pmatrix} \begin{pmatrix} h_1 & \dots & h_1 t_1^m & \dots & h_1 t_1^{s-1} \\ \vdots & & \vdots & & \vdots \\ h_k & \dots & h_k t_k^m & \dots & h_k t_k^{s-1} \\ \vdots & & \vdots & & \vdots \\ h_s & \dots & h_s t_s^m & \dots & h_s t_s^{s-1} \end{pmatrix} \tag{4}$$

$$= VDV^T \tag{5}$$

where V is the Vandermonde matrix given by

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_s \\ \vdots & \vdots & \dots & \vdots \\ t_1^k & t_2^k & \dots & t_s^k \\ \vdots & \vdots & \dots & \vdots \\ t_1^{s-1} & t_2^{s-1} & \dots & t_s^{s-1} \end{pmatrix}$$

and D is a diagonal matrix with $(h_1, \dots, h_s)$ along the main diagonal. Now, note that the Prony matrix will be full ranked if x has *exactly* s non-zero elements.

Motivation for Prony's method

To students familiar with the theory of difference equations, the form in equation (2) is readily recognizable as a solution of a difference equation of order s. This observation was made in [1]. Thus, $\hat{x}$ should satisfy the difference equation

$$\hat{x}_k + \lambda_1 \hat{x}_{k-1} + \dots + \lambda_s \hat{x}_{k-s} = 0 \tag{6}$$

where the corresponding generating polynomial

$$G(z) = 1 + \sum_{k=1}^{s} \lambda_k z^k, \tag{7}$$

must have roots at $t_1, \dots, t_s$. Writing (6) in vector form we have

$$\begin{pmatrix} x_s \\ \vdots \\ x_{s+k} \\ \vdots \\ x_{2s-1} \end{pmatrix} + \begin{pmatrix} \hat{x}_0 & \dots & \hat{x}_m & \dots & \hat{x}_{s-1} \\ \vdots & & \vdots & & \vdots \\ \hat{x}_k & \dots & \hat{x}_{m+k} & \dots & \hat{x}_{k+s-1} \\ \vdots & & \vdots & & \vdots \\ \hat{x}_{s-1} & \dots & \hat{x}_{s+k-1} & \dots & \hat{x}_{2s-2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_s \end{pmatrix} = 0 \tag{8}$$

When the sparsity of x is exactly s, we have established via the Vandermonde Factorization that the matrix in the above equation is invertible. Thus the λ_ks and hence t_ks can be found. By solving a linear system of equations, the h_ks can also be found.

References

[1] A Novel Interpretation of Prony's Method, Pitstick et al. 1998
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