local parametrization $\phi : U \longrightarrow W$, U being an open set in $\mathbf{R}^k$. Then $d\phi : T(U) \longrightarrow T(W)$ is a diffeomorphism. But $T(U) = U \times \mathbf{R}^k$ is an open subset of $\mathbf{R}^{2k}$, so $d\phi$ serves to parametrize the open set $T(W)$ in $T(X)$. Since every point of $T(X)$ sits in such a neighborhood, we have proved

Proposition. The tangent bundle of a manifold is another manifold, and $\dim T(X) = 2 \dim X$.

Now we prove a version of Whitney's result.

Theorem. Every k-dimensional manifold admits a one-to-one immersion in $\mathbf{R}^{2k+1}$.

Proof. In fact, if $X \subset \mathbf{R}^N$ is k-dimensional and $N > 2k + 1$, we shall produce a *linear* projection $\mathbf{R}^N \longrightarrow \mathbf{R}^{2k+1}$ that restricts to a one-to-one immersion of X. Proceeding inductively, we prove that if $f \colon X \longrightarrow \mathbf{R}^M$ is an injective immersion with $M > 2k + 1$, then there exists a unit vector $a \in \mathbf{R}^M$ such that the composition of f with the projection map carrying $\mathbf{R}^M$ onto the orthogonal complement of a is still an injective immersion. Now the complement $H = \{b \in \mathbf{R}^M : b \perp a\}$ is an $M - 1$ dimensional vector subspace of $\mathbf{R}^M$, hence isomorphic to $\mathbf{R}^{M-1}$; thus we obtain an injective immersion into $\mathbf{R}^{M-1}$.

Define a map $h \colon X \times X \times \mathbf{R} \longrightarrow \mathbf{R}^M$ by $h(x, y, t) = t[f(x) - f(y)]$. Also, define a map $g : T(X) \longrightarrow \mathbf{R}^M$ by $g(x, v) = df_x(v)$. Since $M > 2k + 1$, Sard's theorem implies that there exists a point $a \in \mathbf{R}^M$ belonging to neither image; note that $a \neq 0$, since 0 belongs to both images.

Let π be the projection of $\mathbf{R}^M$ onto the orthogonal complement H of a. Certainly $\pi \circ f \colon X \longrightarrow H$ is injective. For suppose that $\pi \circ f(x) = \pi \circ f(y)$. Then the definition of π implies that $f(x) - f(y) = ta$ for some scalar t. If $x \neq y$ then $t \neq 0$, because f is injective. But then $h(x, y, 1/t) = a$, contradicting the choice of a.

Similarly, $\pi \circ f \colon X \longrightarrow H$ is an immersion. For suppose that v is a nonzero vector in $T_x(X)$ for which $d(\pi \circ f)_x(v) = 0$. Because π is linear, the chain rule yields $d(\pi \circ f)_x = \pi \circ df_x$. Thus $\pi \circ df_x(v) = 0$, so $df_x(v) = ta$ for some scalar t. Because f is an immersion, $t \neq 0$. Thus $g(x, 1/t) = a$, again contradicting the choice of a. Q.E.D.

For compact manifolds, one-to-one immersions are the same as embeddings, so we have just proved the embedding theorem in the compact case. In general, we must modify the immersion to make it proper—a topological, not a differential problem. The situation is typical of differential topology; quite often fundamental differential concepts are most naturally and intuitively developed for compact manifolds, then extended by technical tricks to arbitrary manifolds. Rather than allow such technicalities to divert you now