B-**S**plines Approximation

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Apr.27, 2010

Outline

- B-splines and their properties
  – Subspace $\$_{k,\boldsymbol{t}}$

- B-splines approximation
  – Distance from continuous functions to $\$_{k,\boldsymbol{t}}$

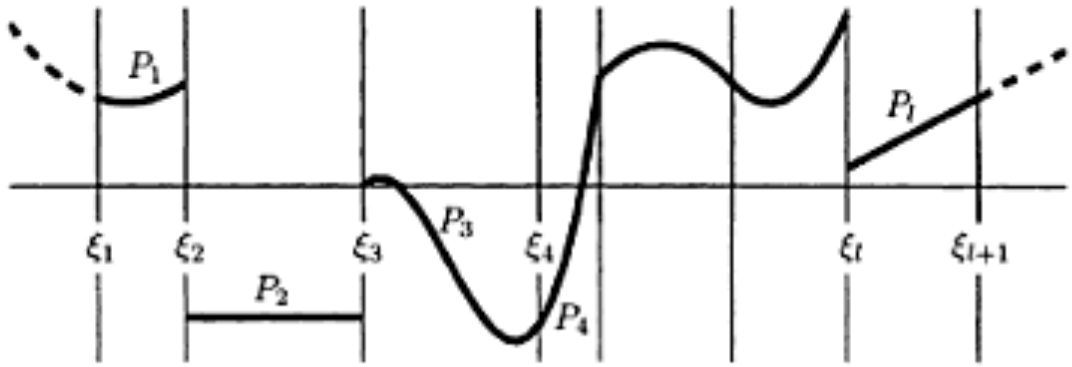
- Knots placement

- Questions?

B-spline—Definition

- Spline
  - A piecewise polynomial function $f(x)$. Let $\xi := (\xi_i)_1^{l+1}$ be a strictly increasing sequence of points, and let k be a positive integer. If $P_1, \ldots, P_l$ is any sequence of l polynomials, each of order k (that is, of degree $<k$), then $f(x)$ of order k is defined as:

$$f(x) := P_i(x) \quad \text{if } \xi_i < x < \xi_{i+1}; i = 1, \ldots, l.$$



B-spline—Definition (Cont'd)

- Linear space $\Pi_{<k,\boldsymbol{\xi}}$
  - Collection of all splines of order *k* with break sequence $\boldsymbol{\xi} = (\xi_i)_1^{l+1}$
  - Dimension is *kl*
- Smoothness constraints
  - $jump_{\xi_i} D^{j-1} f = 0$ for $j = 1,\ldots,v_i$ and $i = 2,\ldots,l$
  - subspace $\Pi_{<k,\boldsymbol{\xi},\mathbf{v}}$ with dim $kl - \sum_{i=2}^{l} v_i$
- Curry and Schoenberg Theorem
  - $\Pi_{<k,\boldsymbol{\xi},\mathbf{v}} \Longleftrightarrow \$_{k,\mathbf{t}} := \left\{ \sum_i \alpha_i B_{i,k,\mathbf{t}} : \alpha_i \text{ real}, \forall i \right\}$

Knots sequence

B-spline—Definition (Cont'd)

- B-splines (Basis splines for $\$_{k,\mathbf{t}}$)
  - Recurrence relation

$$B_{i,k,\mathbf{t}} = \omega_{i,k}(x)B_{i,k-1} + (1-\omega_{i+1,k})B_{i+1,k-1}$$

$$\text{with } \omega_{i,k}(x) := \frac{x - t_i}{t_{i+k-1} - t_i} \text{ and } B_{i,0}(x) = \begin{cases} 1 & \text{if } t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

  - Knots sequence **t**
    - Let *dim* $\$_{k,\mathbf{t}}$ be *n*, then

$$\mathbf{t} = (t_1, t_2, \ldots, t_{n+k}) = (\underbrace{\xi_1, \ldots, \xi_1}_{k}, \ldots, \underbrace{\xi_i, \ldots, \xi_i}_{v_i}, \ldots, \underbrace{\xi_{l+1}, \ldots, \xi_{l+1}}_{k})$$

  - Basis for $\Pi_{<k,\xi,\mathbf{v}}$
    - $f \in \Pi_{<k,\xi,\mathbf{v}}$

B-splines properties

- Local support

$$f \in \$_{<k,\mathbf{t}}, f(x) = \sum_{i=j+1-k}^{j} \alpha_i B_{i,k,\mathbf{t}}(x), \text{ for } x \in [t_j, t_{j+1}]$$

- Non-negativity

$$B_{i,k,\mathbf{t}}(x) \geq 0, \forall i$$

- Partition of unity

$$\sum_{i=j+1-k}^{j} B_{i,k,\mathbf{t}}(x) = 1, \text{ for } x \in [t_j, t_{j+1}]$$

B-splines Approximation

Given $\$_{k,\boldsymbol{t}}$ with knots sequence $\mathbf{t}=(t_1, \ldots, t_{n+k})=(a,\ldots,b)$, choosing $\tau_1 \le \ldots \le \tau_n$ in a way such that τ_i belongs to one and only one knot span, the k^{th} order spline approximation Ag to the continuous function g on $[a .. b]$, is constructed as follows:

$$Ag := \sum_{i=1}^{n} g(\boldsymbol{\tau}_i) B_{i,k,\mathbf{t}} \quad \text{on}\, [a..b]$$

Approximation error

- The approximation error

$$\|g - Ag\| \le C\omega(g;|\mathbf{t}|)$$

with $\|g\| := \max_{a \le x \le b} |g(x)|$, mesh size $|\mathbf{t}| := \max_i \Delta t_i$, the modulus of continuity

$$\omega(g;h) := \max\{|g(x) - g(y)| : |x - y| \le h, x, y \in [a..b]\}$$

- Distance from g to $\$_{k,\mathbf{t}}$

$$dist(g, \$_{k,\mathbf{t}}) := \min\{\|g - s\| : s \in \$_{k,\mathbf{t}}\} \le C\omega(g;|\mathbf{t}|)$$

Approximation error (cont'd)

- Continuous function g

$$g \in C^{(j)}[a..b],\ \operatorname{dist}(g, \$_{<k,\mathbf{t}}) \le C_k |\mathbf{t}|^j \|D^j g\|$$

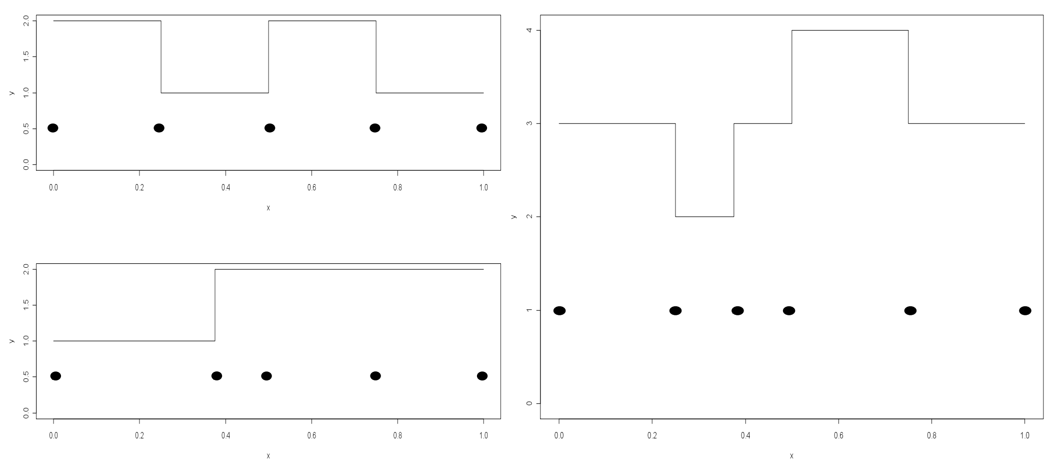
- Local approximation error bound

$$g \in C^{(k)}[a..b],\ \|g - Ag\|_{[t_j..t_{j+1}]} \le C_k |I_j|^k \|D^k g\|_{I_j}$$

with I_j the interval $[t_{j+2-k}..t_{j+k-1}]$, $|I_j|$ the length

Knots placement

- To minimize approximation error in terms of knots is quite difficult
  - $\$_{k,n} := \bigcup \{ f \in \$_{<k,\mathbf{t}} : t_1 = \cdots = t_k = a, t_{n+1} = \cdots = t_{n+k} = b \}$



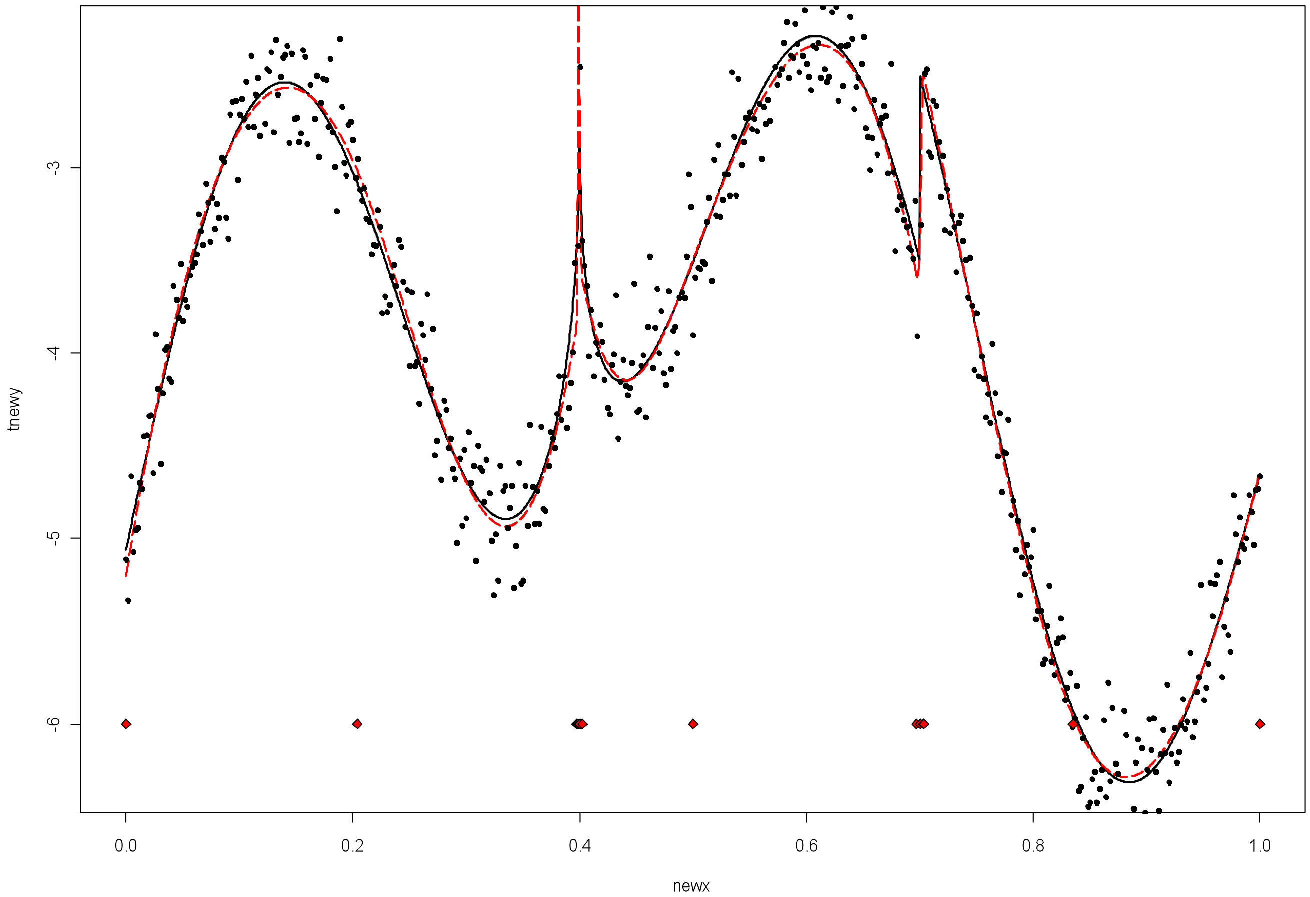
A heuristic way

Idea: specify a group of knots sequences $\mathbf{t}_1,\ldots,\mathbf{t}_N$ such that $\min_{i=1,\ldots,N} C_k|\mathbf{t}_i|^k \|D^k g\| \sim \mathrm{O}(noise)$, define

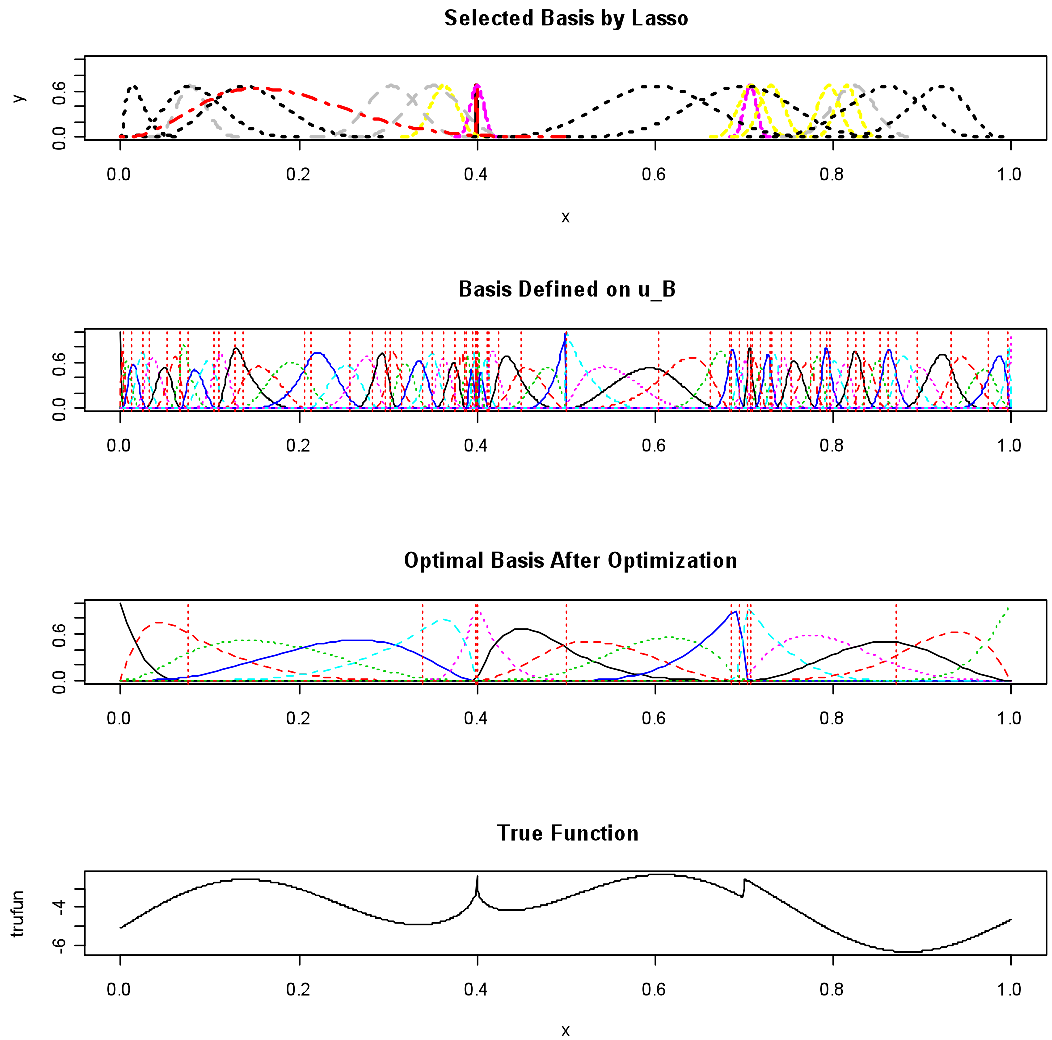
$$Fg := \sum_{r=1}^{N} \sum_{i} \alpha_{r,i} B_{i,k,\mathbf{t}_r}$$

Using variable selection method (Lasso), select optimal bases $optset = \{B_{i,k,\mathbf{t}_r}\}$, and then obtain knots according to the support of each basis in *optset*.

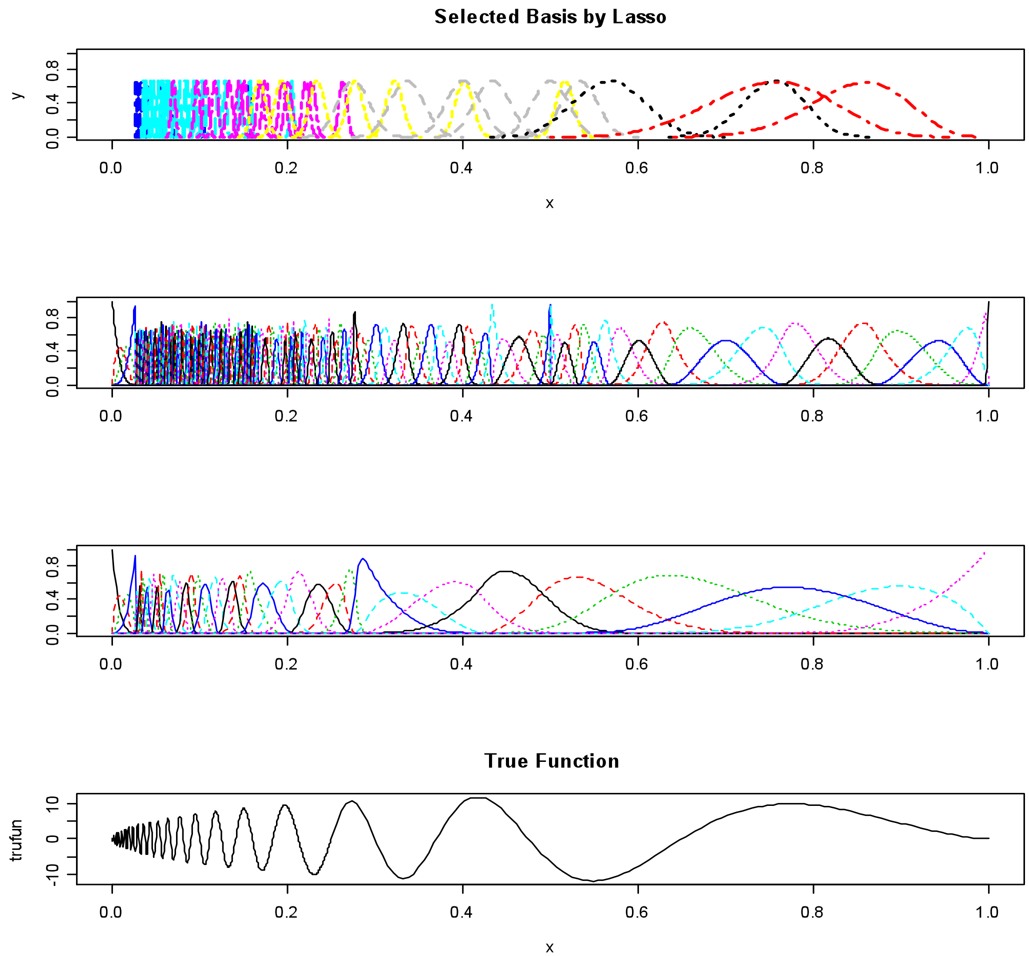
Example



Examples (Cont’d)



Examples (Cont'd)



Questions?