Exact Matrix Completion via Convex Optimization

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Abstract

We consider a problem of considerable practical interest: the recovery of a data matrix from a sampling of its entries. Suppose that we observe m entries selected uniformly at random from a matrix M. Can we complete the matrix and recover the entries that we have not seen?

We show that one can perfectly recover most low-rank matrices from what appears to be an incomplete set of entries. We prove that if the number m of sampled entries obeys

$$m \geq C\, n^{1.2} r \log n$$

for some positive numerical constant C, then with very high probability, most $n \times n$ matrices of rank r can be perfectly recovered by solving a simple convex optimization program. This program finds the matrix with minimum nuclear norm that fits the data. The condition above assumes that the rank is not too large. However, if one replaces the 1.2 exponent with 1.25, then the result holds for all values of the rank. Similar results hold for arbitrary rectangular matrices as well. Our results are connected with the recent literature on compressed sensing, and show that objects other than signals and images can be perfectly reconstructed from very limited information.

Keywords. Matrix completion, low-rank matrices, convex optimization, duality in optimization, nuclear norm minimization, random matrices, noncommutative Khintchine inequality, decoupling, compressed sensing.

1 Introduction

In many practical problems of interest, one would like to recover a matrix from a sampling of its entries. As a motivating example, consider the task of inferring answers in a partially filled out survey. That is, suppose that questions are being asked to a collection of individuals. Then we can form a matrix where the rows index each individual and the columns index the questions. We collect data to fill out this table but unfortunately, many questions are left unanswered. Is it possible to make an educated guess about what the missing answers should be? How can one make such a guess? Formally, we may view this problem as follows. We are interested in recovering a data matrix M with n_1 rows and n_2 columns but only get to observe a number m of its entries which is comparably much smaller than $n_1 n_2$, the total number of entries. Can one recover the matrix M from m of its entries? In general, everyone would agree that this is impossible without some additional information.

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In many instances, however, the matrix we wish to recover is known to be structured in the sense that it is low-rank or approximately low-rank. (We recall for completeness that a matrix with n_1 rows and n_2 columns has rank r if its rows or columns span an r-dimensional space.) Below are two examples of practical scenarios where one would like to be able to recover a low-rank matrix from a sampling of its entries.

- *The Netflix problem.* In the area of recommender systems, users submit ratings on a subset of entries in a database, and the vendor provides recommendations based on the user's preferences [28, 32]. Because users only rate a few items, one would like to infer their preference for unrated items.

  A special instance of this problem is the now famous Netflix problem [2]. Users (rows of the data matrix) are given the opportunity to rate movies (columns of the data matrix) but users typically rate only very few movies so that there are very few scattered observed entries of this data matrix. Yet one would like to complete this matrix so that the vendor (here Netflix) might recommend titles that any particular user is likely to be willing to order. In this case, the data matrix of all user-ratings may be approximately low-rank because it is commonly believed that only a few factors contribute to an individual's tastes or preferences.

- *Triangulation from incomplete data.* Suppose we are given partial information about the distances between objects and would like to reconstruct the low-dimensional geometry describing their locations. For example, we may have a network of low-power wirelessly networked sensors scattered randomly across a region. Suppose each sensor only has the ability to construct distance estimates based on signal strength readings from its nearest fellow sensors. From these noisy distance estimates, we can form a partially observed distance matrix. We can then estimate the true distance matrix whose rank will be equal to two if the sensors are located in a plane or three if they are located in three dimensional space [24, 31]. In this case, we only need to observe a few distances per node to have enough information to reconstruct the positions of the objects.

These examples are of course far from exhaustive and there are many other problems which fall in this general category. For instance, we may have some very limited information about a covariance matrix of interest. Yet, this covariance matrix may be low-rank or approximately low-rank because the variables only depend upon a comparably smaller number of factors.

1.1 Impediments and solutions

Suppose for simplicity that we wish to recover a square $n \times n$ matrix $\boldsymbol{M}$ of rank r.[1] Such a matrix $\boldsymbol{M}$ can be represented by n^2 numbers, but it only has $(2n - r)r$ degrees of freedom. This fact can be revealed by counting parameters in the singular value decomposition (the number of degrees of freedom associated with the description of the singular values and of the left and right singular vectors). When the rank is small, this is considerably smaller than n^2. For instance, when $\boldsymbol{M}$ encodes a 10-dimensional phenomenon, then the number of degrees of freedom is about $20\,n$ offering a reduction in dimensionality by a factor about equal to $n/20$. When n is large (e.g. in the thousands or millions), the data matrix carries much less information than its ambient dimension

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[1]We emphasize that there is nothing special about $\boldsymbol{M}$ being square and all of our discussion would apply to arbitrary rectangular matrices as well. The advantage of focusing on square matrices is a simplified exposition and reduction in the number of parameters of which we need to keep track.

suggests. The problem is now whether it is possible to recover this matrix from a sampling of its entries without having to probe all the n^2 entries, or more generally collect n^2 or more measurements about $\boldsymbol{M}$.

1.1.1 Which matrices?

In general, one cannot hope to be able to recover a low-rank matrix from a sample of its entries. Consider the rank-1 matrix $\boldsymbol{M}$ equal to

$$\boldsymbol{M} = \boldsymbol{e}_1 \boldsymbol{e}_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{1.1}$$

where here and throughout, $\boldsymbol{e}_i$ is the ith canonical basis vector in Euclidean space (the vector with all entries equal to 0 but the ith equal to 1). This matrix has a 1 in the top-right corner and all the other entries are 0. Clearly this matrix cannot be recovered from a sampling of its entries unless we pretty much see all the entries. The reason is that for most sampling sets, we would only get to see zeros so that we would have no way of guessing that the matrix is not zero. For instance, if we were to see 90% of the entries selected at random, then 10% of the time we would only get to see zeroes.

It is therefore impossible to recover *all* low-rank matrices from a set of sampled entries but can one recover *most* of them? To investigate this issue, we introduce a simple model of low-rank matrices. Consider the singular value decomposition (SVD) of a matrix $\boldsymbol{M}$

$$\boldsymbol{M} = \sum_{k=1}^{r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*, \tag{1.2}$$

where the $\boldsymbol{u}_k$'s and $\boldsymbol{v}_k$'s are the left and right singular vectors, and the σ_k's are the singular values (the roots of the eigenvalues of $\boldsymbol{M}^* \boldsymbol{M}$). Then we could think of a *generic* low-rank matrix as follows: the family $\{\boldsymbol{u}_k\}_{1 \le k \le r}$ is selected uniformly at random among all families of r orthonormal vectors, and similarly for the the family $\{\boldsymbol{v}_k\}_{1 \le k \le r}$. The two families may or may not be independent of each other. We make no assumptions about the singular values σ_k. In the sequel, we will refer to this model as the *random orthogonal model*. This model is convenient in the sense that it is both very concrete and simple, and useful in the sense that it will help us fix the main ideas. In the sequel, however, we will consider far more general models. The question for now is whether or not one can recover such a generic matrix from a sampling of its entries.

1.1.2 Which sampling sets?

Clearly, one cannot hope to reconstruct any low-rank matrix $\boldsymbol{M}$—even of rank 1—if the sampling set avoids any column or row of $\boldsymbol{M}$. Suppose that $\boldsymbol{M}$ is of rank 1 and of the form $\boldsymbol{x}\boldsymbol{y}^*$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ so that the (i,j)th entry is given by

$$M_{ij} = x_i y_j.$$

Then if we do not have samples from the first row for example, one could never guess the value of the first component x_1, by any method whatsoever; no information about x_1 is observed. There is

of course nothing special about the first row and this argument extends to any row or column. To have any hope of recovering an unknown matrix, one needs at least one observation per row and one observation per column.

We have just seen that if the sampling is adversarial, e.g. one observes all of the entries of M but those in the first row, then one would not even be able to recover matrices of rank 1. But what happens for most sampling sets? Can one recover a low-rank matrix from almost all sampling sets of cardinality m? Formally, suppose that the set Ω of locations corresponding to the observed entries ($(i,j) \in \Omega$ if M_{ij} is observed) is a set of cardinality m sampled uniformly at random. Then can one recover a generic low-rank matrix M, perhaps with very large probability, from the knowledge of the value of its entries in the set Ω?

1.1.3 Which algorithm?

If the number of measurements is sufficiently large, and if the entries are sufficiently uniformly distributed as above, one might hope that there is only *one* low-rank matrix with these entries. If this were true, one would want to recover the data matrix by solving the optimization problem

$$
\begin{aligned}
& \text{minimize} && \text{rank}(\boldsymbol{X}) \\
& \text{subject to} && X_{ij} = M_{ij} \quad (i,j) \in \Omega,
\end{aligned}
\tag{1.3}
$$

where $\boldsymbol{X}$ is the decision variable and $\text{rank}(\boldsymbol{X})$ is equal to the rank of the matrix $\boldsymbol{X}$. The program (1.3) is a common sense approach which simply seeks the simplest explanation fitting the observed data. If there were only one low-rank object fitting the data, this would recover $\boldsymbol{M}$. This is unfortunately of little practical use because this optimization problem is not only NP-hard, but all known algorithms which provide exact solutions require time doubly exponential in the dimension n of the matrix in both theory and practice [14].

If a matrix has rank r, then it has exactly r nonzero singular values so that the rank function in (1.3) is simply the number of nonvanishing singular values. In this paper, we consider an alternative which minimizes the sum of the singular values over the constraint set. This sum is called the *nuclear norm*,

$$
\|\boldsymbol{X}\|_* = \sum_{k=1}^{n} \sigma_k(\boldsymbol{X})
\tag{1.4}
$$

where, here and below, $\sigma_k(\boldsymbol{X})$ denotes the kth largest singular value of $\boldsymbol{X}$. The heuristic optimization is then given by

$$
\begin{aligned}
& \text{minimize} && \|\boldsymbol{X}\|_* \\
& \text{subject to} && X_{ij} = M_{ij} \quad (i,j) \in \Omega.
\end{aligned}
\tag{1.5}
$$

Whereas the rank function counts the number of nonvanishing singular values, the nuclear norm sums their amplitude and in some sense, is to the rank functional what the convex ℓ_1 norm is to the counting ℓ_0 norm in the area of sparse signal recovery. The main point here is that the nuclear norm is a convex function and, as we will discuss in Section 1.4 can be optimized efficiently via semidefinite programming.

1.1.4 A first typical result

Our first result shows that, perhaps unexpectedly, this heuristic optimization recovers a generic $\boldsymbol{M}$ when the number of randomly sampled entries is large enough. We will prove the following:

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Theorem 1.1 *Let M be an $n_1 \times n_2$ matrix of rank r sampled from the random orthogonal model, and put $n = \max(n_1, n_2)$. Suppose we observe m entries of M with locations sampled uniformly at random. Then there are numerical constants C and c such that if*

$$m \geq C\, n^{5/4} r \log n\,, \tag{1.6}$$

the minimizer to the problem (1.5) is unique and equal to M with probability at least $1 - cn^{-3}$; that is to say, the semidefinite program (1.5) recovers all the entries of M with no error. In addition, if $r \leq n^{1/5}$, then the recovery is exact with probability at least $1 - cn^{-3}$ provided that

$$m \geq C\, n^{6/5} r \log n\,. \tag{1.7}$$

The theorem states that a surprisingly small number of entries are sufficient to complete a generic low-rank matrix. For small values of the rank, e.g. when $r = O(1)$ or $r = O(\log n)$, one only needs to see on the order of $n^{6/5}$ entries (ignoring logarithmic factors) which is considerably smaller than n^2—the total number of entries of a squared matrix. The real feat, however, is that the recovery algorithm is tractable and very concrete. Hence the contribution is twofold:

- Under the hypotheses of Theorem 1.1, there is a unique low-rank matrix which is consistent with the observed entries.

- Further, this matrix can be recovered by the convex optimization (1.5). In other words, for most problems, the nuclear norm relaxation is *formally equivalent* to the combinatorially hard rank minimization problem (1.3).

Theorem 1.1 is in fact a special instance of a far more general theorem that covers a much larger set of matrices M. We describe this general class of matrices and precise recovery conditions in the next section.

1.2   Main results

As seen in our first example (1.1), it is impossible to recover a matrix which is equal to zero in nearly all of its entries unless we see all the entries of the matrix. To recover a low-rank matrix, this matrix cannot be in the null space of the sampling operator giving the values of a subset of the entries. Now it is easy to see that if the singular vectors of a matrix M are highly concentrated, then M could very well be in the null-space of the sampling operator. For instance consider the rank-2 symmetric matrix M given by

$$M = \sum_{k=1}^{2} \sigma_k u_k u_k^*, \qquad \begin{aligned} u_1 &= (e_1 + e_2)/\sqrt{2}, \\ u_2 &= (e_1 - e_2)/\sqrt{2}, \end{aligned}$$

where the singular values are arbitrary. Then this matrix vanishes everywhere except in the top-left 2×2 corner and one would basically need to see all the entries of M to be able to recover this matrix exactly by any method whatsoever. There is an endless list of examples of this sort. Hence, we arrive at the notion that, somehow, the singular vectors need to be sufficiently spread—that is, uncorrelated with the standard basis—in order to minimize the number of observations needed to recover a low-rank matrix.[2] This motivates the following definition.

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[2]Both the left and right singular vectors need to be uncorrelated with the standard basis. Indeed, the matrix $e_1 v^*$ has its first row equal to v and all the others equal to zero. Clearly, this rank-1 matrix cannot be recovered unless we basically see all of its entries.

Definition 1.2 *Let U be a subspace of $\mathbb{R}^n$ of dimension r and $\boldsymbol{P}_U$ be the orthogonal projection onto U. Then the coherence of U (vis-à-vis the standard basis $(\boldsymbol{e}_i)$) is defined to be*

$$\mu(U) \equiv \frac{n}{r} \max_{1 \leq i \leq n} \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2. \tag{1.8}$$

Note that for any subspace, the smallest $\mu(U)$ can be is 1, achieved, for example, if U is spanned by vectors whose entries all have magnitude $1/\sqrt{n}$. The largest possible value for $\mu(U)$ is n/r which would correspond to any subspace that contains a standard basis element. We shall be primarily interested in subspace with low coherence as matrices whose column and row spaces have low coherence cannot really be in the null space of the sampling operator. For instance, we will see that the random subspaces discussed above have nearly minimal coherence.

To state our main result, we introduce two assumptions about an $n_1 \times n_2$ matrix $\boldsymbol{M}$ whose SVD is given by $\boldsymbol{M} = \sum_{1 \leq k \leq r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*$ and with column and row spaces denoted by U and V respectively.

A0 The coherences obey $\max(\mu(U), \mu(V)) \leq \mu_0$ for some positive μ_0.

A1 The $n_1 \times n_2$ matrix $\sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^*$ has a maximum entry bounded by $\mu_1 \sqrt{r/(n_1 n_2)}$ in absolute value for some positive μ_1.

The μ's above may depend on r and n_1, n_2. Moreover, note that **A1** always holds with $\mu_1 = \mu_0 \sqrt{r}$ since the (i,j)th entry of the matrix $\sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^*$ is given by $\sum_{1 \leq k \leq r} u_{ik} v_{jk}$ and by the Cauchy-Schwarz inequality,

$$\left| \sum_{1 \leq k \leq r} u_{ik} v_{jk} \right| \leq \sqrt{\sum_{1 \leq k \leq r} |u_{ik}|^2} \sqrt{\sum_{1 \leq k \leq r} |v_{jk}|^2} \leq \frac{\mu_0 r}{\sqrt{n_1 n_2}}.$$

Hence, for sufficiently small ranks, μ_1 is comparable to μ_0. As we will see in Section 2, for larger ranks, both subspaces selected from the uniform distribution and spaces constructed as the span of singular vectors with bounded entries are not only incoherent with the standard basis, but also obey **A1** with high probability for values of μ_1 at most logarithmic in n_1 and/or n_2. Below we will assume that μ_1 is greater than or equal to 1.

We are in the position to state our main result: if a matrix has row and column spaces that are incoherent with the standard basis, then nuclear norm minimization can recover this matrix from a random sampling of a small number of entries.

Theorem 1.3 *Let $\boldsymbol{M}$ be an $n_1 \times n_2$ matrix of rank r obeying **A0** and **A1** and put $n = \max(n_1, n_2)$. Suppose we observe m entries of $\boldsymbol{M}$ with locations sampled uniformly at random. Then there exist constants C, c such that if*

$$m \geq C \max(\mu_1^2, \mu_0^{1/2} \mu_1, \mu_0 n^{1/4}) \, nr(\beta \log n) \tag{1.9}$$

for some $\beta > 2$, then the minimizer to the problem (1.5) is unique and equal to $\boldsymbol{M}$ with probability at least $1 - cn^{-\beta}$. For $r \leq \mu_0^{-1} n^{1/5}$ this estimate can be improved to

$$m \geq C \, \mu_0 \, n^{6/5} r(\beta \log n) \tag{1.10}$$

with the same probability of success.

Theorem 1.3 asserts that if the coherence is low, few samples are required to recover $\boldsymbol{M}$. For example, if $\mu_0 = O(1)$ and the rank is not too large, then the recovery is exact with large probability provided that

$$m \geq C\, n^{6/5} r \log n. \tag{1.11}$$

We give two illustrative examples of matrices with incoherent column and row spaces. This list is by no means exhaustive.

1. The first example is the random orthogonal model. For values of the rank r greater than $\log n$, $\mu(U)$ and $\mu(V)$ are $O(1)$, $\mu_1 = O(\log n)$ both with very large probability. Hence, the recovery is exact provided that m obeys (1.6) or (1.7). Specializing Theorem 1.3 to these values of the parameters gives Theorem 1.1. Hence, Theorem 1.1 is a special case of our general recovery result.

2. The second example is more general and, in a nutshell, simply requires that the components of the singular vectors of $\boldsymbol{M}$ are small. Assume that the $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$'s obey

$$\max_{ij} |\langle \boldsymbol{e}_i, \boldsymbol{u}_j \rangle|^2 \leq \mu_B/n, \quad \max_{ij} |\langle \boldsymbol{e}_i, \boldsymbol{v}_j \rangle|^2 \leq \mu_B/n, \tag{1.12}$$

for some value of $\mu_B = O(1)$. Then the maximum coherence is at most μ_B since $\mu(U) \leq \mu_B$ and $\mu(V) \leq \mu_B$. Further, we will see in Section 2 that **A1** holds most of the time with $\mu_1 = O(\sqrt{\log n})$. Thus, for matrices with singular vectors obeying (1.12), the recovery is exact provided that m obeys (1.11) for values of the rank not exceeding $\mu_B^{-1} n^{1/5}$.

1.3 Extensions

Our main result (Theorem 1.3) extends to a variety of other low-rank matrix completion problems beyond the sampling of entries. Indeed, suppose we have two orthonormal bases $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ and $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n$ of $\mathbb{R}^n$, and that we are interested in solving the rank minimization problem

$$\begin{array}{ll} \text{minimize} & \text{rank}(\boldsymbol{X}) \\ \text{subject to} & \boldsymbol{f}_i^* \boldsymbol{X} \boldsymbol{g}_j = \boldsymbol{f}_i^* \boldsymbol{M} \boldsymbol{g}_j, \quad (i,j) \in \Omega, \end{array} \tag{1.13}$$

This comes up in a number of applications. As a motivating example, there has been a great deal of interest in the machine learning community in developing specialized algorithms for the *multiclass* and *multitask* learning problems (see, e.g., [1, 3, 5]). In multiclass learning, the goal is to build multiple classifiers with the same training data to distinguish between more than two categories. For example, in face recognition, one might want to classify whether an image patch corresponds to an eye, nose, or mouth. In multitask learning, we have a large set of data, but have a variety of different classification tasks, and, for each task, only partial subsets of the data are relevant. For instance, in activity recognition, we may have acquired sets of observations of multiple subjects and want to determine if each observed person is walking or running. However, a different classifier is to be learned for each individual, and it is not clear how having access to the full collection of observations can improve classification performance. Multitask learning aims precisely to take advantage of the access to the full database to improve performance on the individual tasks.

In the abstract formulation of this problem for linear classifiers, we have K classes to distinguish and are given training examples $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$. For each example, we are given partial labeling information about which classes it belongs or does not belong to. That is, for each example $\boldsymbol{f}_j$

and class k, we may either be told that $\boldsymbol{f}_j$ belongs to class k, be told $\boldsymbol{f}_j$ does not belong to class k, or provided no information about the membership of $\boldsymbol{f}_j$ to class k. For each class $1 \leq k \leq K$, we would like to produce a linear function $\boldsymbol{w}_k$ such that $\boldsymbol{w}_k^* \boldsymbol{f}_i > 0$ if $\boldsymbol{f}_i$ belongs to class k and $\boldsymbol{w}_k^* \boldsymbol{f}_i < 0$ otherwise. Formally, we can search for the vector $\boldsymbol{w}_k$ that satisfies the equality constraints $\boldsymbol{w}_k^* \boldsymbol{f}_i = y_{ik}$ where $y_{ik} = 1$ if we are told that $\boldsymbol{f}_i$ belongs to class k, $y_{ik} = -1$ if we are told that $\boldsymbol{f}_i$ does not belong to class k, and y_{ik} unconstrained if we are not provided information. A common hypothesis in the multitask setting is that the $\boldsymbol{w}_k$ corresponding to each of the classes together span a very low dimensional subspace with dimension significantly smaller than K [1,3,5]. That is, the basic assumption is that

$$\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]$$

is low-rank. Hence, the multiclass learning problem can be cast as (1.13) with observations of the form $\boldsymbol{f}_i^* \boldsymbol{W} \boldsymbol{e}_j$.

To see that our theorem provides conditions under which (1.13) can be solved via nuclear norm minimization, note that there exist unitary transformations $\boldsymbol{F}$ and $\boldsymbol{G}$ such that $\boldsymbol{e}_j = \boldsymbol{F} \boldsymbol{f}_j$ and $\boldsymbol{e}_j = \boldsymbol{G} \boldsymbol{g}_j$ for each $j = 1, \ldots, n$. Hence,

$$\boldsymbol{f}_i^* \boldsymbol{X} \boldsymbol{g}_j = \boldsymbol{e}_i^* (\boldsymbol{F} \boldsymbol{X} \boldsymbol{G}^*) \boldsymbol{e}_j.$$

Then if the conditions of Theorem 1.3 hold for the matrix $\boldsymbol{F} \boldsymbol{X} \boldsymbol{G}^*$, it is immediate that nuclear norm minimization finds the unique optimal solution of (1.13) when we are provided a large enough random collection of the inner products $\boldsymbol{f}_i^* \boldsymbol{M} \boldsymbol{g}_j$. In other words, all that is needed is that the column and row spaces of $\boldsymbol{M}$ be respectively incoherent with the basis $(\boldsymbol{f}_i)$ and $(\boldsymbol{g}_i)$.

From this perspective, we additionally remark that our results likely extend to the case where one observes a small number of arbitrary linear functionals of a hidden matrix $\boldsymbol{M}$. Set $N = n^2$ and $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N$ be an orthonormal basis for the linear space of $n \times n$ matrices with the usual inner product $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \mathrm{trace}(\boldsymbol{X}^* \boldsymbol{Y})$. Then we expect our results should also apply to the rank minimization problem

$$\begin{aligned}
\text{minimize} \quad & \mathrm{rank}(\boldsymbol{X}) \\
\text{subject to} \quad & \langle \boldsymbol{A}_k, \boldsymbol{X} \rangle = \langle \boldsymbol{A}_k, \boldsymbol{M} \rangle \quad k \in \Omega,
\end{aligned} \tag{1.14}$$

where $\Omega \subset \{1, \ldots, N\}$ is selected uniformly at random. In fact, (1.14) is (1.3) when the orthobasis is the canonical basis $(\boldsymbol{e}_i \boldsymbol{e}_j^*)_{1 \leq i,j \leq n}$. Here, those low-rank matrices which have small inner product with all the basis elements $\boldsymbol{A}_k$ may be recoverable by nuclear norm minimization. To avoid unnecessary confusion and notational clutter, we leave this general low-rank recovery problem for future work.

1.4 Connections, alternatives and prior art

Nuclear norm minimization is a recent heuristic introduced by Fazel in [18], and is an extension of the trace heuristic often used by the control community, see e.g. [6, 26]. Indeed, when the matrix variable is symmetric and positive semidefinite, the nuclear norm of $\boldsymbol{X}$ is the sum of the (nonnegative) eigenvalues and thus equal to the trace of $\boldsymbol{X}$. Hence, for positive semidefinite unknowns, (1.5) would simply minimize the trace over the constraint set:

$$\begin{aligned}
\text{minimize} \quad & \mathrm{trace}(\boldsymbol{X}) \\
\text{subject to} \quad & X_{ij} = M_{ij} \quad (i,j) \in \Omega \ . \\
& \boldsymbol{X} \succeq 0
\end{aligned}$$

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This is a semidefinite program. Even for the general matrix $\boldsymbol{M}$ which may not be positive definite or even symmetric, the nuclear norm heuristic can be formulated in terms of semidefinite programming as, for instance, the program (1.5) is equivalent to

$$
\begin{aligned}
\text{minimize} \quad & \text{trace}(\boldsymbol{W}_1) + \text{trace}(\boldsymbol{W}_2) \\
\text{subject to} \quad & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\
& \begin{bmatrix} \boldsymbol{W}_1 & \boldsymbol{X} \\ \boldsymbol{X}^* & \boldsymbol{W}_2 \end{bmatrix} \succeq 0
\end{aligned}
$$

with optimization variables $\boldsymbol{X}$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$, (see, e.g., [18,35]). There are many efficient algorithms and high-quality software available for solving these types of problems.

Our work is inspired by results in the emerging field of *compressive sampling* or *compressed sensing*, a new paradigm for acquiring information about objects of interest from what appears to be a highly incomplete set of measurements [11, 13, 17]. In practice, this means for example that high-resolution imaging is possible with fewer sensors, or that one can speed up signal acquisition time in biomedical applications by orders of magnitude, simply by taking far fewer specially coded samples. Mathematically speaking, we wish to reconstruct a signal $\boldsymbol{x} \in \mathbb{R}^n$ from a small number measurements $\boldsymbol{y} = \Phi\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^m$, and m is much smaller than n; i.e. we have far fewer equations than unknowns. In general, one cannot hope to reconstruct $\boldsymbol{x}$ but assume now that the object we wish to recover is known to be structured in the sense that it is sparse (or approximately sparse). This means that the unknown object depends upon a smaller number of unknown parameters. Then it has been shown that ℓ_1 minimization allows recovery of sparse signals from remarkably few measurements: supposing Φ is chosen randomly from a suitable distribution, then with very high probability, all sparse signals with about k nonzero entries can be recovered from on the order of $k \log n$ measurements. For instance, if $\boldsymbol{x}$ is k-sparse in the Fourier domain, i.e. $\boldsymbol{x}$ is a superposition of k sinusoids, then it can be perfectly recovered with high probability—by ℓ_1 minimization—from the knowledge of about $k \log n$ of its entries sampled uniformly at random [11].

From this viewpoint, the results in this paper greatly extend the theory of compressed sensing by showing that other types of interesting objects or structures, beyond sparse signals and images, can be recovered from a limited set of measurements. Moreover, the techniques for proving our main results build upon ideas from the compressed sensing literature together with probabilistic tools such as the powerful techniques of Bourgain and of Rudelson for bounding norms of operators between Banach spaces.

Our notion of incoherence generalizes the concept of the same name in compressive sampling. Notably, in [10], the authors introduce the notion of the incoherence of a unitary transformation. Letting $\boldsymbol{U}$ be an $n \times n$ unitary matrix, the *coherence* of $\boldsymbol{U}$ is given by

$$
\mu(\boldsymbol{U}) = n \max_{j,k} |U_{jk}|^2.
$$

This quantity ranges in values from 1 for a unitary transformation whose entries all have the same magnitude to n for the identity matrix. Using this notion, [10] showed that with high probability, a k-sparse signal could be recovered via linear programming from the observation of the inner product of the signal with $m = \Omega(\mu(\boldsymbol{U})k \log n)$ randomly selected columns of the matrix $\boldsymbol{U}$. This result provided a generalization of the celebrated results about partial Fourier observations described in [11], a special case where $\mu(\boldsymbol{U}) = 1$. This paper generalizes the notion of incoherence to problems beyond the setting of sparse signal recovery.

In [27], the authors studied the nuclear norm heuristic applied to a related problem where partial information about a matrix M is available from m equations of the form

$$\langle \boldsymbol{A}^{(k)}, M \rangle = \sum_{ij} A_{ij}^{(k)} M_{ij} = b_k, \qquad k = 1, \ldots, m, \qquad (1.15)$$

where for each k, $\{A_{ij}^{(k)}\}_{ij}$ is an i.i.d. sequence of Gaussian or Bernoulli random variables and the sequences $\{\boldsymbol{A}^{(k)}\}$ are also independent from each other (the sequences $\{\boldsymbol{A}^{(k)}\}$ and $\{b_k\}$ are available to the analyst). Building on the concept of *restricted isometry* introduced in [12] in the context of sparse signal recovery, [27] establishes the first sufficient conditions for which the nuclear norm heuristic returns the minimum rank element in the constraint set. They prove that the heuristic succeeds with large probability whenever the number m of available measurements is greater than a constant times $2nr \log n$ for $n \times n$ matrices. Although this is an interesting result, a serious impediment to this approach is that one needs to essentially measure random projections of the unknown data matrix—a situation which unfortunately does not commonly arise in practice. Further, the measurements in (1.15) give some information about *all* the entries of M whereas in our problem, information about most of the entries is simply not available. In particular, the results and techniques introduced in [27] do not begin to address the matrix completion problem of interest to us in this paper. As a consequence, our methods are completely different; for example, they do not rely on any notions of restricted isometry. Instead, as we discuss below, we prove the existence of a Lagrange multiplier for the optimization (1.5) that certifies the unique optimal solution is precisely the matrix that we wish to recover.

Finally, we would like to briefly discuss the possibility of other recovery algorithms when the sampling happens to be chosen in a very special fashion. For example, suppose that M is generic and that we precisely observe every entry in the first r rows and columns of the matrix. Write M in block form as

$$M = \left[\begin{array}{cc} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{array} \right]$$

with $\boldsymbol{M}_{11}$ an $r \times r$ matrix. In the special case that $\boldsymbol{M}_{11}$ is invertible and M has rank r, then it is easy to verify that $\boldsymbol{M}_{22} = \boldsymbol{M}_{21} \boldsymbol{M}_{11}^{-1} \boldsymbol{M}_{12}$. One can prove this identity by forming the SVD of M, for example. That is, if M is generic, and the upper $r \times r$ block is invertible, and we observe *every* entry in the first r rows and columns, we can recover M. This result immediately generalizes to the case where one observes precisely r rows and r columns and the $r \times r$ matrix at the intersection of the observed rows and columns is invertible. However, this scheme has many practical drawbacks that stand in the way of a generalization to a completion algorithm from a general set of entries. First, if we miss *any* entry in these rows or columns, we cannot recover M, nor can we leverage any information provided by entries of $\boldsymbol{M}_{22}$. Second, if the matrix has rank less than r, and we observe r rows and columns, a combinatorial search to find the collection that has an invertible square sub-block is required. Moreover, because of the matrix inversion, the algorithm is rather fragile to noise in the entries.

1.5 Notations and organization of the paper

The paper is organized as follows. We first argue in Section 2 that the random orthogonal model and, more generally, matrices with incoherent column and row spaces obey the assumptions of the general Theorem 1.3. To prove Theorem 1.3, we first establish sufficient conditions which guarantee

that the true low-rank matrix $\boldsymbol{M}$ is the unique solution to (1.5) in Section 3. One of these conditions is the existence of a dual vector obeying two crucial properties. Section 4 constructs such a dual vector and provides the overall architecture of the proof which shows that, indeed, this vector obeys the desired properties provided that the number of measurements is sufficiently large. Surprisingly, as explored in Section 5, the existence of a dual vector certifying that $\boldsymbol{M}$ is unique is related to some problems in random graph theory including "the coupon collector's problem." Following this discussion, we prove our main result via several intermediate results which are all proven in Section 6. Section 7 introduces numerical experiments showing that matrix completion based on nuclear norm minimization works well in practice. Section 8 closes the paper with a short summary of our findings, a discussion of important extensions and improvements. In particular, we will discuss possible ways of improving the 1.2 exponent in (1.10) so that it gets closer to 1. Finally, the Appendix provides proofs of auxiliary lemmas supporting our main argument.

Before continuing, we provide here a brief summary of the notations used throughout the paper. Matrices are bold capital, vectors are bold lowercase and scalars or entries are not bold. For instance, $\boldsymbol{X}$ is a matrix and X_{ij} its (i,j)th entry. Likewise $\boldsymbol{x}$ is a vector and x_i its ith component. When we have a collection of vectors $\boldsymbol{u}_k \in \mathbb{R}^n$ for $1 \le k \le d$, we will denote by u_{ik} the ith component of the vector $\boldsymbol{u}_k$ and $[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d]$ will denote the $n \times d$ matrix whose kth column is $\boldsymbol{u}_k$.

A variety of norms on matrices will be discussed. The spectral norm of a matrix is denoted by $\|\boldsymbol{X}\|$. The Euclidean inner product between two matrices is $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \text{trace}(\boldsymbol{X}^*\boldsymbol{Y})$, and the corresponding Euclidean norm, called the Frobenius or Hilbert-Schmidt norm, is denoted $\|\boldsymbol{X}\|_F$. That is, $\|\boldsymbol{X}\|_F = \langle \boldsymbol{X}, \boldsymbol{X} \rangle^{1/2}$. The nuclear norm of a matrix $\boldsymbol{X}$ is $\|\boldsymbol{X}\|_*$. The maximum entry of $\boldsymbol{X}$ (in absolute value) is denoted by $\|\boldsymbol{X}\|_\infty \equiv \max_{ij} |X_{ij}|$. For vectors, we will only consider the usual Euclidean ℓ_2 norm which we simply write as $\|\boldsymbol{x}\|$.

Further, we will also manipulate linear transformation which acts on matrices and will use caligraphic letters for these operators as in $\mathcal{A}(\boldsymbol{X})$. In particular, the identity operator will be denoted by $\mathcal{I}$. The only norm we will consider for these operators is their spectral norm (the top singular value) denoted by $\|\mathcal{A}\| = \sup_{\boldsymbol{X}:\|\boldsymbol{X}\|_F \le 1} \|\mathcal{A}(\boldsymbol{X})\|_F$.

Finally, we adopt the convention that C denotes a numerical constant independent of the matrix dimensions, rank, and number of measurements, whose value may change from line to line. Certain special constants with precise numerical values will be ornamented with subscripts (e.g., C_R). Any exceptions to this notational scheme will be noted in the text.

2   Which matrices are incoherent?

In this section we restrict our attention to square $n \times n$ matrices, but the extension to rectangular $n_1 \times n_2$ matrices immediately follows by setting $n = \max(n_1, n_2)$.

2.1   Incoherent bases span incoherent subspaces

Almost all $n \times n$ matrices $\boldsymbol{M}$ with singular vectors $\{\boldsymbol{u}_k\}_{1 \le k \le r}$ and $\{\boldsymbol{v}_k\}_{1 \le k \le r}$ obeying the size property (1.12) also satisfy the assumptions **A0** and **A1** with $\mu_0 = \mu_B$, $\mu_1 = C\mu_B\sqrt{\log n}$ for some positive constant C. As mentioned above, **A0** holds automatically, but, observe that **A1** would not hold with a small value of μ_1 if two rows of the matrices $[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r]$ and $[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r]$ are identical

with all entries of magnitude $\sqrt{\mu_B/n}$ since it is not hard to see that in this case

$$\|\sum_k \boldsymbol{u}_k \boldsymbol{v}_k^*\|_\infty = \mu_B \, r/n.$$

Certainly, this example is constructed in a very special way, and should occur infrequently. We now show that it is generically unlikely.

Consider the matrix

$$\sum_{k=1}^r \epsilon_k \boldsymbol{u}_k \boldsymbol{v}_k^*, \tag{2.1}$$

where $\{\epsilon_k\}_{1 \le k \le r}$ is an arbitrary sign sequence. For almost all choices of sign sequences, **A1** is satisfied with $\mu_1 = O(\mu_B \sqrt{\log n})$. Indeed, if one selects the signs uniformly at random, then for each $\beta > 0$,

$$\mathbb{P}(\|\sum_{k=1}^r \epsilon_k \boldsymbol{u}_k \boldsymbol{v}_k\|_\infty \ge \mu_B \, \sqrt{8\beta r \log n}/n) \le (2n^2)\, n^{-\beta}. \tag{2.2}$$

This is of interest because suppose the low-rank matrix we wish to recover is of the form

$$\boldsymbol{M} = \sum_{k=1}^r \lambda_k \boldsymbol{u}_k \boldsymbol{v}_k^* \tag{2.3}$$

with scalars λ_k. Since the vectors $\{\boldsymbol{u}_k\}$ and $\{\boldsymbol{v}_k\}$ are orthogonal, the singular values of $\boldsymbol{M}$ are given by $|\lambda_k|$ and the singular vectors are given by $\mathrm{sgn}(\lambda_k)\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ for $k = 1, \ldots, r$. Hence, in this model **A1** concerns the maximum entry of the matrix given by (2.1) with $\epsilon_k = \mathrm{sgn}(\lambda_k)$. That is to say, for most sign patterns, the matrix of interest obeys an appropriate size condition. We emphasize here that the only thing that we assumed about the $\boldsymbol{u}_k$'s and $\boldsymbol{v}_k$'s was that they had small entries. In particular, they could be equal to each other as would be the case for a symmetric matrix.

The claim (2.2) is a simple application of Hoeffding's inequality. The (i, j)th entry of (2.1) is given by

$$Z_{ij} = \sum_{1 \le k \le r} \epsilon_k u_{ik} v_{jk},$$

and is a sum of r zero-mean independent random variables, each bounded by μ_B/n. Therefore,

$$\mathbb{P}(|Z_{ij}| \ge \lambda \mu_B \sqrt{r}/n) \le 2e^{-\lambda^2/8}.$$

Setting λ proportional to $\sqrt{\log n}$ and applying the union bound gives the claim.

To summarize, we say that $\boldsymbol{M}$ is sampled from the *incoherent basis model* if it is of the form

$$\boldsymbol{M} = \sum_{k=1}^r \epsilon_k \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*; \tag{2.4}$$

$\{\epsilon_k\}_{1 \le k \le r}$ is a random sign sequence, and $\{\boldsymbol{u}_k\}_{1 \le k \le r}$ and $\{\boldsymbol{v}_k\}_{1 \le k \le r}$ have maximum entries of size at most $\sqrt{\mu_B/n}$.

Lemma 2.1 *There exist numerical constants c and C such that for any $\beta > 0$, matrices from the incoherent basis model obey the assumption* **A1** *with $\mu_1 \le C \mu_B \sqrt{(\beta + 2) \log n}$ with probability at least $1 - cn^{-\beta}$.*

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2.2 Random subspaces span incoherent subspaces

In this section, we prove that the random orthogonal model obeys the two assumptions **A0** and **A1** (with appropriate values for the μ's) with large probability.

Lemma 2.2 *Set $\bar{r} = \max(r, \log n)$. Then there exist constants C and c such that the random orthogonal model obeys:*[3]

1. $\max_i \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2 \leq C\,\bar{r}/n$,

2. $\|\sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^*\|_\infty \leq C\log n\,\sqrt{\bar{r}}/n$.

with probability $1 - cn^{-3}\log n$.

We note that an argument similar to the following proof would give that if C of the form $K\beta$ where K is a fixed numerical constant, we can achieve a probability at least $1 - cn^{-\beta}$ provided that n is sufficiently large. To establish these facts, we make use of the standard result below [21].

Lemma 2.3 *Let Y_d be distributed as a chi-squared random variable with d degrees of freedom. Then for each $t > 0$*

$$\mathbb{P}(Y_d - d \geq t\sqrt{2d} + t^2) \leq e^{-t^2/2} \quad and \quad \mathbb{P}(Y_d - d \leq -t\sqrt{2d}) \leq e^{-t^2/2}. \tag{2.5}$$

We will use (2.5) as follows: for each $\epsilon \in (0,1)$ we have

$$\mathbb{P}(Y_d \geq d\,(1-\epsilon)^{-1}) \leq e^{-\epsilon^2 d/4} \quad and \quad \mathbb{P}(Y_d \leq d\,(1-\epsilon)) \leq e^{-\epsilon^2 d/4}. \tag{2.6}$$

We begin with the second assertion of Lemma 2.2 since it will imply the first as well. Observe that it follows from

$$\|\boldsymbol{P}_U \boldsymbol{e}_i\|^2 = \sum_{1 \leq k \leq r} u_{ik}^2, \tag{2.7}$$

that $Z_r \equiv \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2$ (a is fixed) is the squared Euclidean length of the first r components of a unit vector uniformly distributed on the unit sphere in n dimensions. Now suppose that $x_1, x_2, \ldots, x_n$ are i.i.d. $N(0,1)$. Then the distribution of a unit vector uniformly distributed on the sphere is that of $\boldsymbol{x}/\|\boldsymbol{x}\|$ and, therefore, the law of Z_r is that of Y_r/Y_n, where $Y_r = \sum_{k \leq r} x_k^2$. Fix $\epsilon > 0$ and consider the event $A_{n,\epsilon} = \{Y_n/n \geq 1 - \epsilon\}$. For each $\lambda > 0$, it follows from (2.6) that

$$
\begin{aligned}
\mathbb{P}(Z_r - r/n \geq \lambda\sqrt{2r}/n) &= \mathbb{P}(Y_r \geq [r + \lambda\sqrt{2r}]Y_n/n) \\
&\leq \mathbb{P}(Y_r \geq [r + \lambda\sqrt{2r}]Y_n/n \text{ and } A_{n,\epsilon}) + \mathbb{P}(A_{n,\epsilon}^c) \\
&\leq \mathbb{P}(Y_r \geq [r + \lambda\sqrt{2r}][1 - \epsilon]) + e^{-\epsilon^2 n/4} \\
&= \mathbb{P}(Y_r - r \geq \lambda\sqrt{2r}[1 - \epsilon - \epsilon\sqrt{r/2\lambda^2}]) + e^{-\epsilon^2 n/4}.
\end{aligned}
$$

Now pick $\epsilon = 4(n^{-1}\log n)^{1/2}$, $\lambda = 8\sqrt{2\log n}$ and assume that n is sufficiently large so that

$$\epsilon(1 + \sqrt{r/2\lambda^2}) \leq 1/2.$$

---

[3]When $r \geq C'(\log n)^3$ for some positive constant C', a better estimate is possible, namely, $\|\sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^*\|_\infty \leq C\sqrt{r\log n}/n$.

Then
$$\mathbb{P}(Z_r - r/n \ge \lambda\sqrt{2r}/n) \le \mathbb{P}(Y_r - r \ge (\lambda/2)\sqrt{2r}) + n^{-4}.$$

Assume now that $r \ge 4\log n$ (which means that $\lambda \le 4\sqrt{2r}$). Then it follows from (2.5) that

$$\mathbb{P}(Y_r - r \ge (\lambda/2)\sqrt{2r}) \le \mathbb{P}(Y_r - r \ge (\lambda/4)\sqrt{2r} + (\lambda/4)^2) \le e^{-\lambda^2/32} = n^{-4}.$$

Hence

$$\mathbb{P}(Z_r - r/n \ge 16\sqrt{r\log n}/n) \le 2n^{-4}$$

and, therefore,

$$\mathbb{P}(\max_i \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2 - r/n \ge 16\sqrt{r\log n}/n) \le 2n^{-3} \tag{2.8}$$

by the union bound. Note that (2.8) establishes the first claim of the lemma (even for $r < 4\log n$ since in this case $Z_r \le Z_{\lceil 4\log n\rceil}$).

It remains to establish the second claim. Notice that by symmetry, $\boldsymbol{E} = \sum_{1\le k\le r} \boldsymbol{u}_k \boldsymbol{v}_k^*$ has the same distribution as

$$\boldsymbol{F} = \sum_{k=1}^r \epsilon_k \boldsymbol{u}_k \boldsymbol{v}_k^*,$$

where $\{\epsilon_k\}$ is an independent Rademacher sequence. It then follows from Hoeffding's inequality that conditional on $\{\boldsymbol{u}_k\}$ and $\{\boldsymbol{v}_k\}$ we have

$$\mathbb{P}(|F_{ij}| > t) \le 2e^{-t^2/2\sigma_{ij}^2}, \quad \sigma_{ij}^2 = \sum_{1\le k\le r} u_{ik}^2 v_{ik}^2.$$

Our previous results indicate that $\max_{ij} |v_{ij}|^2 \le (10\log n)/n$ with large probability and thus

$$\sigma_{ij}^2 \le 10\,\frac{\log n}{n}\,\|\boldsymbol{P}_U \boldsymbol{e}_i\|^2.$$

Set $\bar{r} = \max(r, \log n)$. Since $\|\boldsymbol{P}_U \boldsymbol{e}_i\|^2 \le C\bar{r}/n$ with large probability, we have

$$\sigma_{ij}^2 \le C(\log n)\,\bar{r}/n^2$$

with large probability. Hence the marginal distribution of F_{ij} obeys

$$\mathbb{P}(|F_{ij}| > \lambda\sqrt{\bar{r}}/n) \le 2e^{-\gamma\lambda^2/\log n} + \mathbb{P}(\sigma_{ij}^2 \ge C(\log n)\bar{r}/n^2).$$

for some numerical constant γ. Picking $\lambda = \gamma'\log n$ where γ' is a sufficiently large numerical constant gives

$$\|\boldsymbol{F}\|_\infty \le C\,(\log n)\,\sqrt{\bar{r}}/n$$

with large probability. Since $\boldsymbol{E}$ and $\boldsymbol{F}$ have the same distribution, the second claim follows.

The claim about the size of $\max_{ij} |v_{ij}|^2$ is straightforward since our techniques show that for each $\lambda > 0$

$$\mathbb{P}(Z_1 \ge \lambda(\log n)/n) \le \mathbb{P}(Y_1 \ge \lambda(1-\epsilon)\log n) + e^{-\epsilon^2 n/4}.$$

Moreover,

$$\mathbb{P}(Y_1 \ge \lambda(1-\epsilon)\log n) = \mathbb{P}(|x_1| \ge \sqrt{\lambda(1-\epsilon)\log n}) \le 2e^{-\frac{1}{2}\lambda(1-\epsilon)\log n}.$$

If n is sufficiently large so that $\epsilon \le 1/5$, this gives $\mathbb{P}(Z_1 \ge 10(\log n)/n) \le 3n^{-4}$ and, therefore,

$$\mathbb{P}(\max_{ij} |v_{ij}|^2 \ge 10(\log n)/n) \le 12n^{-3}\log n$$

since the maximum is taken over at most $4n\log n$ pairs.

3 Duality

Let $\mathcal{R}_\Omega : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{|\Omega|}$ be the sampling operator which extracts the observed entries, $\mathcal{R}_\Omega(\boldsymbol{X}) = (X_{ij})_{ij \in \Omega}$, so that the constraint in (1.5) becomes $\mathcal{R}_\Omega(\boldsymbol{X}) = \mathcal{R}_\Omega(\boldsymbol{M})$. Standard convex optimization theory asserts that $\boldsymbol{X}$ is solution to (1.5) if there exists a dual vector (or Lagrange multiplier) $\lambda \in \mathbb{R}^{|\Omega|}$ such that $\mathcal{R}_\Omega^* \lambda$ is a *subgradient* of the nuclear norm at the point $\boldsymbol{X}$, which we denote by

$$\mathcal{R}_\Omega^* \lambda \in \partial \|\boldsymbol{X}\|_* \tag{3.1}$$

(see, e.g. [7]). Recall the definition of a subgradient of a convex function $f : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}$. We say that $\boldsymbol{Y}$ is a subgradient of f at $\boldsymbol{X}_0$, denoted $\boldsymbol{Y} \in \partial f(\boldsymbol{X}_0)$, if

$$f(\boldsymbol{X}) \geq f(\boldsymbol{X}_0) + \langle \boldsymbol{Y}, \boldsymbol{X} - \boldsymbol{X}_0 \rangle \tag{3.2}$$

for all $\boldsymbol{X}$.

Suppose $\boldsymbol{X}_0 \in \mathbb{R}^{n_1 \times n_2}$ has rank r with a singular value decomposition given by

$$\boldsymbol{X}_0 = \sum_{1 \leq k \leq r} \sigma_k \, \boldsymbol{u}_k \boldsymbol{v}_k^*, \tag{3.3}$$

With these notations, $\boldsymbol{Y}$ is a subgradient of the nuclear norm at $\boldsymbol{X}_0$ if and only if it is of the form

$$\boldsymbol{Y} = \sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^* + \boldsymbol{W}, \tag{3.4}$$

where $\boldsymbol{W}$ obeys the following two properties:

(i) the column space of $\boldsymbol{W}$ is orthogonal to $U \equiv \mathrm{span}\,(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$, and the row space of $\boldsymbol{W}$ is orthogonal to $V \equiv \mathrm{span}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$;

(ii) the spectral norm of $\boldsymbol{W}$ is less than or equal to 1.

(see, e.g., [23,36]). To express these properties concisely, it is convenient to introduce the orthogonal decomposition $\mathbb{R}^{n_1 \times n_2} = T \oplus T^\perp$ where T is the linear space spanned by elements of the form $\boldsymbol{u}_k \boldsymbol{x}^*$ and $\boldsymbol{y} \boldsymbol{v}_k^*$, $1 \leq k \leq r$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are arbitrary, and $T^\perp$ is its orthogonal complement. Note that $\dim(T) = r(n_1 + n_2 - r)$, precisely the number of degrees of freedom in the set of $n_1 \times n_2$ matrices of rank r. $T^\perp$ is the subspace of matrices spanned by the family $(\boldsymbol{x}\boldsymbol{y}^*)$, where $\boldsymbol{x}$ (respectively $\boldsymbol{y}$) is any vector orthogonal to U (respectively V).

The orthogonal projection $\mathcal{P}_T$ onto T is given by

$$\mathcal{P}_T(\boldsymbol{X}) = \boldsymbol{P}_U \boldsymbol{X} + \boldsymbol{X} \boldsymbol{P}_V - \boldsymbol{P}_U \boldsymbol{X} \boldsymbol{P}_V, \tag{3.5}$$

where $\boldsymbol{P}_U$ and $\boldsymbol{P}_V$ are the orthogonal projections onto U and V. Note here that while $\boldsymbol{P}_U$ and $\boldsymbol{P}_V$ are matrices, $\mathcal{P}_T$ is a linear operator mapping matrices to matrices. We also have

$$\mathcal{P}_{T^\perp}(\boldsymbol{X}) = (\mathcal{I} - \mathcal{P}_T)(\boldsymbol{X}) = (\boldsymbol{I}_{n_1} - \boldsymbol{P}_U)\boldsymbol{X}(\boldsymbol{I}_{n_2} - \boldsymbol{P}_V)$$

where $\boldsymbol{I}_d$ denotes the $d \times d$ identity matrix. With these notations, $\boldsymbol{Y} \in \partial \|\boldsymbol{X}_0\|_*$ if

(i') $\mathcal{P}_T(\boldsymbol{Y}) = \sum_{1 \leq k \leq r} \boldsymbol{u}_k \boldsymbol{v}_k^*$,

(ii') and $\|\mathcal{P}_{T^\perp} \boldsymbol{Y}\| \leq 1$.

Now that we have characterized the subgradient of the nuclear norm, the lemma below gives sufficient conditions for the uniqueness of the minimizer to (1.5).

Lemma 3.1 *Consider a matrix* $\boldsymbol{X}_0 = \sum_{k=1}^{r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*$ *of rank* r *which is feasible for the problem* (1.5), *and suppose that the following two conditions hold:*

1. there exists a dual point λ *such that* $\boldsymbol{Y} = \mathcal{R}_\Omega^* \lambda$ *obeys*

$$\mathcal{P}_T(\boldsymbol{Y}) = \sum_{k=1}^{r} \boldsymbol{u}_k \boldsymbol{v}_k^*, \qquad \|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1; \tag{3.6}$$

2. the sampling operator $\mathcal{R}_\Omega$ *restricted to elements in* T *is injective.*

Then $\boldsymbol{X}_0$ *is the unique minimizer.*

Before proving this result, we would like to emphasize that this lemma provides a clear strategy for proving our main result, namely, Theorem 1.3. Letting $\boldsymbol{M} = \sum_{k=1}^{r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*$, $\boldsymbol{M}$ is the unique solution to (1.5) if the injectivity condition holds and if one can find a dual point λ such that $\boldsymbol{Y} = \mathcal{R}_\Omega^* \lambda$ obeys (3.6).

The proof of Lemma 3.1 uses a standard fact which states that the nuclear norm and the spectral norm are dual to one another.

Lemma 3.2 *For each pair* $\boldsymbol{W}$ *and* $\boldsymbol{H}$, *we have*

$$\langle \boldsymbol{W}, \boldsymbol{H} \rangle \leq \|\boldsymbol{W}\| \, \|\boldsymbol{H}\|_*.$$

In addition, for each $\boldsymbol{H}$, *there is a* $\boldsymbol{W}$ *obeying* $\|\boldsymbol{W}\| = 1$ *which achieves the equality.*

A variety of proofs are available for this Lemma, and an elementary argument is sketched in [27]. We now turn to the proof of Lemma 3.1.

Proof [of Lemma 3.1] Consider any perturbation $\boldsymbol{X}_0 + \boldsymbol{H}$ where $\mathcal{R}_\Omega(\boldsymbol{H}) = 0$. Then for any $\boldsymbol{W}^0$ obeying (i)–(ii), $\sum_{k=1}^{r} \boldsymbol{u}_k \boldsymbol{v}_k^* + \boldsymbol{W}^0$ is a subgradient of the nuclear norm at X_0 and, therefore,

$$\|\boldsymbol{X}_0 + \boldsymbol{H}\|_* \geq \|\boldsymbol{X}_0\|_* + \langle \sum_{k=1}^{r} \boldsymbol{u}_k \boldsymbol{v}_k^* + \boldsymbol{W}^0, \boldsymbol{H} \rangle.$$

Letting $\boldsymbol{W} = \mathcal{P}_{T^\perp}(\boldsymbol{Y})$, we may write $\sum_{k=1}^{r} \boldsymbol{u}_k \boldsymbol{v}_k^* = \mathcal{R}_\Omega^* \lambda - \boldsymbol{W}$. Since $\|\boldsymbol{W}\| < 1$ and $\mathcal{R}_\Omega(\boldsymbol{H}) = 0$, it then follows that

$$\|\boldsymbol{X}_0 + \boldsymbol{H}\|_* \geq \|\boldsymbol{X}_0\|_* + \langle \boldsymbol{W}^0 - \boldsymbol{W}, \boldsymbol{H} \rangle.$$

Now by construction

$$\langle \boldsymbol{W}^0 - \boldsymbol{W}, \boldsymbol{H} \rangle = \langle \mathcal{P}_{T^\perp}(\boldsymbol{W}^0 - \boldsymbol{W}), \boldsymbol{H} \rangle = \langle \boldsymbol{W}^0 - \boldsymbol{W}, \mathcal{P}_{T^\perp}(\boldsymbol{H}) \rangle.$$

We use Lemma 3.2 and set $\boldsymbol{W}^0 = \mathcal{P}_{T^\perp}(\boldsymbol{Z})$ where $\boldsymbol{Z}$ is any matrix obeying $\|\boldsymbol{Z}\| \leq 1$ and $\langle \boldsymbol{Z}, \mathcal{P}_{T^\perp}(\boldsymbol{H}) \rangle = \|\mathcal{P}_{T^\perp}(\boldsymbol{H})\|_*$. Then $\boldsymbol{W}^0 \in T^\perp$, $\|\boldsymbol{W}^0\| \leq 1$, and

$$\langle \boldsymbol{W}^0 - \boldsymbol{W}, \boldsymbol{H} \rangle \geq (1 - \|\boldsymbol{W}\|) \, \|\mathcal{P}_{T^\perp}(\boldsymbol{H})\|_*,$$

which by assumption is strictly positive unless $\mathcal{P}_{T^\perp}(\boldsymbol{H}) = 0$. In other words, $\|\boldsymbol{X}_0 + \boldsymbol{H}\|_* > \|\boldsymbol{X}_0\|_*$ unless $\mathcal{P}_{T^\perp}(\boldsymbol{H}) = 0$. Assume then that $\mathcal{P}_{T^\perp}(\boldsymbol{H}) = 0$ or equivalently that $\boldsymbol{H} \in T$. Then $\mathcal{R}_\Omega(\boldsymbol{H}) = 0$ implies that $\boldsymbol{H} = 0$ by the injectivity assumption. In conclusion, $\|\boldsymbol{X}_0 + \boldsymbol{H}\|_* > \|\boldsymbol{X}\|_*$ unless $\boldsymbol{H} = 0$. ∎

4   Architecture of the proof

Our strategy to prove that $\boldsymbol{M} = \sum_{1 \le k \le r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*$ is the unique minimizer to (1.5) is to construct a matrix $\boldsymbol{Y}$ which vanishes on Ω^c and obeys the conditions of Lemma 3.1 (and show the injectivity of the sampling operator restricted to matrices in T along the way). Set $\mathcal{P}_\Omega$ to be the orthogonal projector onto the indices in Ω so that the (i,j)th component of $\mathcal{P}_\Omega(\boldsymbol{X})$ is equal to X_{ij} if $(i,j) \in \Omega$ and zero otherwise. Our candidate $\boldsymbol{Y}$ will be the solution to

$$
\begin{aligned}
&\text{minimize} && \|\boldsymbol{X}\|_F \\
&\text{subject to} && (\mathcal{P}_T \mathcal{P}_\Omega)(\boldsymbol{X}) = \sum_{k=1}^r \boldsymbol{u}_k \boldsymbol{v}_k^*.
\end{aligned} \tag{4.1}
$$

The matrix $\boldsymbol{Y}$ vanishes on Ω^c as otherwise it would not be an optimal solution since $\mathcal{P}_\Omega(\boldsymbol{Y})$ would obey the constraint and have a smaller Frobenius norm. Hence $\boldsymbol{Y} = \mathcal{P}_\Omega(\boldsymbol{Y})$ and $\mathcal{P}_T(\boldsymbol{Y}) = \sum_{k=1}^r \boldsymbol{u}_k \boldsymbol{v}_k^*$. Since the Pythagoras formula gives

$$
\begin{aligned}
\|\boldsymbol{Y}\|_F^2 = \|\mathcal{P}_T(\boldsymbol{Y})\|_F^2 + \|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\|_F^2 &= \|\sum_{k=1}^r \boldsymbol{u}_k \boldsymbol{v}_k^*\|_F^2 + \|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\|_F^2 \\
&= r + \|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\|_F^2,
\end{aligned}
$$

minimizing the Frobenius norm of $\boldsymbol{X}$ amounts to minimizing the Frobenius norm of $\mathcal{P}_{T^\perp}(\boldsymbol{X})$ under the constraint $\mathcal{P}_T(\boldsymbol{X}) = \sum_{k=1}^r \boldsymbol{u}_k \boldsymbol{v}_k^*$. Our motivation is twofold. First, the solution to the least-squares problem (4.1) has a closed form that is amenable to analysis. Second, by forcing $\mathcal{P}_{T^\perp}(\boldsymbol{Y})$ to be small in the Frobenius norm, we hope that it will be small in the spectral norm as well, and establishing that $\|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1$ would prove that $\boldsymbol{M}$ is the unique solution to (1.5).

To compute the solution to (4.1), we introduce the operator $\mathcal{A}_{\Omega T}$ defined by

$$
\mathcal{A}_{\Omega T}(\boldsymbol{M}) = \mathcal{P}_\Omega \mathcal{P}_T(\boldsymbol{M}).
$$

Then, if $\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T} = \mathcal{P}_T \mathcal{P}_\Omega \mathcal{P}_T$ has full rank when restricted to T, the minimizer to (4.1) is given by

$$
\boldsymbol{Y} = \mathcal{A}_{\Omega T}(\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T})^{-1}(\boldsymbol{E}), \qquad \boldsymbol{E} \equiv \sum_{k=1}^r \boldsymbol{u}_k \boldsymbol{v}_k^*. \tag{4.2}
$$

We clarify the meaning of (4.2) to avoid any confusion. $(\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T})^{-1}(\boldsymbol{E})$ is meant to be that element $\boldsymbol{F}$ in T obeying $(\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T})(\boldsymbol{F}) = \boldsymbol{E}$.

To summarize the aims of our proof strategy,

- We must first show that $\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T} = \mathcal{P}_T \mathcal{P}_\Omega \mathcal{P}_T$ is a one-to-one linear mapping from T onto itself. In this case, $\mathcal{A}_{\Omega T} = \mathcal{P}_\Omega \mathcal{P}_T$—as a mapping from T to $\mathbb{R}^{n_1 \times n_2}$—is injective. This is the second sufficient condition of Lemma 3.1. Moreover, our ansatz for $\boldsymbol{Y}$ given by (4.2) is well-defined.

- Having established that $\boldsymbol{Y}$ is well-defined, we will show that

$$
\|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1,
$$

thus proving the first sufficient condition.

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4.1 The Bernoulli model

Instead of showing that the theorem holds when Ω is a set of size m sampled uniformly at random, we prove the theorem for a subset Ω' sampled according to the *Bernoulli model*. Here and below, $\{\delta_{ij}\}_{1 \le i \le n_1, 1 \le j \le n_2}$ is a sequence of independent identically distributed 0/1 Bernoulli random variables with

$$\mathbb{P}(\delta_{ij} = 1) = p \equiv \frac{m}{n_1 n_2}, \tag{4.3}$$

and define

$$\Omega' = \{(i, j) : \delta_{ij} = 1\}. \tag{4.4}$$

Note that $\mathbb{E}\,|\Omega'| = m$, so that the average cardinality of Ω' is that of Ω. Then following the same reasoning as the argument developed in Section II.C of [11] shows that the probability of 'failure' under the uniform model is bounded by 2 times the probability of failure under the Bernoulli model; the failure event is the event on which the solution to (1.5) is not exact. Hence, we can restrict our attention to the Bernoulli model and from now on, we will assume that Ω is given by (4.4). This is advantageous because the Bernoulli model admits a simpler analysis than uniform sampling thanks to the independence between the δ_{ij}'s.

4.2 The injectivity property

We study the injectivity of $\mathcal{A}_{\Omega T}$, which also shows that $\boldsymbol{Y}$ is well-defined. To prove this, we will show that the linear operator $p^{-1}\mathcal{P}_T(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{P}_T$ has small operator norm, which we recall is $\sup_{\|\boldsymbol{X}\|_F \le 1} p^{-1}\|\mathcal{P}_T(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{P}_T(\boldsymbol{X})\|_F$.

Theorem 4.1 *Suppose Ω is sampled according to the Bernoulli model (4.3)–(4.4) and put $n = \max(n_1, n_2)$. Suppose that the coherences obey $\max(\mu(U), \mu(V)) \le \mu_0$. Then, there is a numerical constants C_R such that for all $\beta > 1$,*

$$p^{-1}\,\|\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T - p\mathcal{P}_T\| \le C_R\,\sqrt{\frac{\mu_0\,nr(\beta\log n)}{m}} \tag{4.5}$$

with probability at least $1 - 3n^{-\beta}$ provided that $C_R\,\sqrt{\frac{\mu_0\,nr(\beta\log n)}{m}} < 1$.

Proof Decompose any matrix $\boldsymbol{X}$ as $\boldsymbol{X} = \sum_{ab}\langle\boldsymbol{X}, \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle\boldsymbol{e}_a\boldsymbol{e}_b^*$ so that

$$\mathcal{P}_T(\boldsymbol{X}) = \sum_{ab}\langle\mathcal{P}_T(\boldsymbol{X}), \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle\boldsymbol{e}_a\boldsymbol{e}_b^* = \sum_{ab}\langle\boldsymbol{X}, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle\boldsymbol{e}_a\boldsymbol{e}_b^*.$$

Hence, $\mathcal{P}_\Omega\mathcal{P}_T(\boldsymbol{X}) = \sum_{ab}\delta_{ab}\,\langle\boldsymbol{X}, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle\,\boldsymbol{e}_a\boldsymbol{e}_b^*$ which gives

$$(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)(\boldsymbol{X}) = \sum_{ab}\delta_{ab}\,\langle\boldsymbol{X}, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle\,\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*).$$

In other words,

$$\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T = \sum_{ab}\delta_{ab}\,\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) \otimes \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*).$$

It follows from the definition (3.5) of $\mathcal{P}_T$ that

$$\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) = (\boldsymbol{P}_U\boldsymbol{e}_a)\boldsymbol{e}_b^* + \boldsymbol{e}_a(\boldsymbol{P}_V\boldsymbol{e}_b)^* - (\boldsymbol{P}_U\boldsymbol{e}_a)(\boldsymbol{P}_V\boldsymbol{e}_b)^*. \tag{4.6}$$

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This gives

$$\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 = \langle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*), \boldsymbol{e}_a\boldsymbol{e}_b^* \rangle = \|\boldsymbol{P}_U\boldsymbol{e}_a\|^2 + \|\boldsymbol{P}_V\boldsymbol{e}_b\|^2 - \|\boldsymbol{P}_U\boldsymbol{e}_a\|^2\,\|\boldsymbol{P}_V\boldsymbol{e}_b\|^2 \qquad (4.7)$$

and since $\|\boldsymbol{P}_U\boldsymbol{e}_a\|^2 \le \mu(U)r/n_1$ and $\|\boldsymbol{P}_V\boldsymbol{e}_b\|^2 \le \mu(U)r/n_2$,

$$\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 \le 2\mu_0 r/\min(n_1, n_2). \qquad (4.8)$$

Now the fact that the operator $\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ does not deviate from its expected value

$$\mathbb{E}(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T) = \mathcal{P}_T(\mathbb{E}\,\mathcal{P}_\Omega)\mathcal{P}_T = \mathcal{P}_T(p\mathcal{I})\mathcal{P}_T = p\mathcal{P}_T$$

in the spectral norm is related to Rudelson's selection theorem [29]. The first part of the theorem below may be found in [10] for example, see also [30] for a very similar statement.

Theorem 4.2 *[10] Let $\{\delta_{ab}\}$ be independent 0/1 Bernoulli variables with $\mathbb{P}(\delta_{ab} = 1) = p = \frac{m}{n_1 n_2}$ and put $n = \max(n_1, n_2)$. Suppose that $\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 \le 2\mu_0 r/n$. Set*

$$Z \equiv p^{-1}\|\sum_{ab}(\delta_{ab} - p)\,\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) \otimes \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\| = p^{-1}\|\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T - p\mathcal{P}_T\|.$$

1. There exists a constant C_R' such that

$$\mathbb{E}\,Z \le C_R'\sqrt{\frac{\mu_0\,nr\,\log n}{m}} \qquad (4.9)$$

provided that the right-hand side is smaller than 1.

2. Suppose $\mathbb{E}\,Z \le 1$. Then for each $\lambda > 0$, we have

$$\mathbb{P}\left(|Z - \mathbb{E}\,Z| > \lambda\sqrt{\frac{\mu_0\,nr\,\log n}{m}}\right) \le 3\exp\left(-\gamma_0'\,\min\left\{\lambda^2\log n, \lambda\sqrt{\frac{m\log n}{\mu_0\,nr}}\right\}\right) \qquad (4.10)$$

for some positive constant γ_0'.

As mentioned above, the first part, namely, (4.9) is an application of an established result which states that if $\{y_i\}$ is a family of vectors in $\mathbb{R}^d$ and $\{\delta_i\}$ is a 0/1 Bernoulli sequence with $\mathbb{P}(\delta_i = 1) = p$, then

$$p^{-1}\|\sum_i(\delta_i - p)y_i \otimes y_i\| \le C\sqrt{\frac{\log d}{p}}\max_i\|y_i\|$$

for some $C > 0$ provided that the right-hand side is less than 1. The proof may be found in the cited literature, e.g. in [10]. Hence, the first part follows from applying this result to vectors of the form $\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)$ and using the available bound on $\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F$. The second part follows from Talagrand's concentration inequality and may be found in the Appendix.

Set $\lambda = \sqrt{\beta/\gamma_0'}$ and assume that $m > (\beta/\gamma_0')\mu_0\,nr\log n$. Then the left-hand side of (4.10) is bounded by $3n^{-\beta}$ and thus, we established that

$$Z \le C_R'\sqrt{\frac{\mu_0\,nr\,\log n}{m}} + \frac{1}{\sqrt{\gamma_0'}}\sqrt{\frac{\mu_0\,nr\,\beta\log n}{m}}$$

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with probability at least $1 - 3n^{-\beta}$. Setting $C_R = C_R' + 1/\sqrt{\gamma_0'}$ finishes the proof. ∎

Take m large enough so that $C_R \sqrt{\mu_0 (nr/m) \log n} \leq 1/2$. Then it follows from (4.5) that

$$\frac{p}{2}\|\mathcal{P}_T(\boldsymbol{X})\|_F \leq \|(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)(\boldsymbol{X})\|_F \leq \frac{3p}{2}\|\mathcal{P}_T(\boldsymbol{X})\|_F \tag{4.11}$$

for all $\boldsymbol{X}$ with large probability. In particular, the operator $\mathcal{A}_{\Omega T}^* \mathcal{A}_{\Omega T} = \mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ mapping T onto itself is well-conditioned and hence invertible. An immediate consequence is the following:

Corollary 4.3 *Assume that $C_R \sqrt{\mu_0 nr(\log n)/m} \leq 1/2$. With the same probability as in Theorem 4.1, we have*

$$\|\mathcal{P}_\Omega\mathcal{P}_T(\boldsymbol{X})\|_F \leq \sqrt{3p/2}\|\mathcal{P}_T(\boldsymbol{X})\|_F. \tag{4.12}$$

Proof We have $\|\mathcal{P}_\Omega\mathcal{P}_T(\boldsymbol{X})\|_F^2 = \langle \boldsymbol{X}, (\mathcal{P}_\Omega\mathcal{P}_T)^*(\mathcal{P}_\Omega\mathcal{P}_T)\boldsymbol{X}\rangle = \langle \boldsymbol{X}, (\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)\boldsymbol{X}\rangle$ and thus

$$\|\mathcal{P}_\Omega\mathcal{P}_T(\boldsymbol{X})\|_F^2 = \langle \mathcal{P}_T\boldsymbol{X}, (\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)\boldsymbol{X}\rangle \leq \|\mathcal{P}_T(\boldsymbol{X})\|_F \|(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)(\boldsymbol{X})\|_F,$$

where the inequality is due to Cauchy-Schwarz. The conclusion (4.12) follows from (4.11). ∎

4.3 The size property

In this section, we explain how we will show that $\|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1$. This result will follow from five lemmas that we will prove in Section 6. Introduce

$$\mathcal{H} \equiv \mathcal{P}_T - p^{-1}\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T,$$

which obeys $\|\mathcal{H}(\boldsymbol{X})\|_F \leq C_R \sqrt{\mu_0(nr/m)\beta \log n}\|\mathcal{P}_T(\boldsymbol{X})\|_F$ with large probability because of Theorem 4.1. For any matrix $\boldsymbol{X} \in T$, $(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)^{-1}(\boldsymbol{X})$ can be expressed in terms of the power series

$$(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)^{-1}(\boldsymbol{X}) = p^{-1}(\boldsymbol{X} + \mathcal{H}(\boldsymbol{X}) + \mathcal{H}^2(\boldsymbol{X}) + \dots)$$

for $\mathcal{H}$ is a contraction when m is sufficiently large. Since $\boldsymbol{Y} = \mathcal{P}_\Omega\mathcal{P}_T(\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T)^{-1}(\sum_{1\leq k\leq r} \boldsymbol{u}_k\boldsymbol{v}_k^*)$, $\mathcal{P}_{T^\perp}(\boldsymbol{Y})$ may be decomposed as

$$\mathcal{P}_{T^\perp}(\boldsymbol{Y}) = p^{-1}(\mathcal{P}_{T^\perp}\mathcal{P}_\Omega\mathcal{P}_T)(\boldsymbol{E} + \mathcal{H}(\boldsymbol{E}) + \mathcal{H}^2(\boldsymbol{E}) + \dots), \qquad \boldsymbol{E} = \sum_{1\leq k\leq r} \boldsymbol{u}_k\boldsymbol{v}_k^*. \tag{4.13}$$

To bound the norm of the left-hand side, it is of course sufficient to bound the norm of the summands in the right-hand side. Taking the following five lemmas together establishes Theorem 1.3.

Lemma 4.4 *Fix $\beta \geq 2$ and $\lambda \geq 1$. There is a numerical constant C_0 such that if $m \geq \lambda\mu_1^2 nr\beta \log n$, then*

$$p^{-1}\|(\mathcal{P}_{T^\perp}\mathcal{P}_\Omega\mathcal{P}_T)\boldsymbol{E}\| \leq C_0 \lambda^{-1/2}. \tag{4.14}$$

with probability at least $1 - n^{-\beta}$.

Lemma 4.5 *Fix $\beta \geq 2$ and $\lambda \geq 1$. There are numerical constants C_1 and c_1 such that if $m \geq \lambda\mu_1 \max(\sqrt{\mu_0}, \mu_1) nr\beta \log n$, then*

$$p^{-1}\|(\mathcal{P}_{T^\perp}\mathcal{P}_\Omega\mathcal{P}_T)\mathcal{H}(\boldsymbol{E})\| \leq C_1 \lambda^{-1} \tag{4.15}$$

with probability at least $1 - c_1 n^{-\beta}$.

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Lemma 4.6 *Fix $\beta \geq 2$ and $\lambda \geq 1$. There are numerical constants C_2 and c_2 such that if $m \geq \lambda \mu_0^{4/3} \, nr^{4/3} \beta \log n$, then*

$$p^{-1} \left\| (\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \mathcal{H}^2(\boldsymbol{E}) \right\| \leq C_2 \, \lambda^{-3/2} \tag{4.16}$$

with probability at least $1 - c_2 n^{-\beta}$.

Lemma 4.7 *Fix $\beta \geq 2$ and $\lambda \geq 1$. There are numerical constants C_3 and c_3 such that if $m \geq \lambda \mu_0^2 \, nr^2 \beta \log n$, then*

$$p^{-1} \left\| (\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \mathcal{H}^3(\boldsymbol{E}) \right\| \leq C_3 \, \lambda^{-1/2} \tag{4.17}$$

with probability at least $1 - c_3 n^{-\beta}$.

Lemma 4.8 *Under the assumptions of Theorem 4.1, there is a numerical constant C_{k_0} such that if $m \geq (2C_R)^2 \mu_0 nr\beta \log n$, then*

$$p^{-1} \left\| (\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E}) \right\| \leq C_{k_0} \left(\frac{n^2 r}{m} \right)^{1/2} \left(\frac{\mu_0 nr\beta \log n}{m} \right)^{k_0/2} \tag{4.18}$$

with probability at least $1 - n^{-\beta}$.

Let us now show how we may combine these lemmas to prove our main results. Under all of the assumptions of Theorem 1.3, consider the four Lemmas 4.4, 4.5, 4.6 and 4.8, the latter applied with $k_0 = 3$. Together they imply that there are numerical constants c and C such that $\|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1$ with probability at least $1 - cn^{-\beta}$ provided that the number of samples obeys

$$m \geq C \, \max(\mu_1^2, \mu_0^{1/2}\mu_1, \mu_0^{4/3}r^{1/3}, \mu_0 n^{1/4}) \, nr\beta \log n \tag{4.19}$$

for some constant C. The four expressions in the maximum come from Lemmas 4.4, 4.5, 4.6 and 4.8 in this order. Now the bound (4.19) is only interesting in the range when $\mu_0 n^{1/4} r$ is smaller than a constant times n as otherwise the right-hand side is greater than n^2 (this would say that one would see all the entries in which case our claim is trivial). When $\mu_0 r \leq n^{3/4}$, $(\mu_0 r)^{4/3} \leq \mu_0 n^{5/4} r$ and thus the recovery is exact provided that m obeys (1.9).

For the case concerning small values of the rank, we consider all five lemmas and apply Lemma 4.8, the latter applied with $k_0 = 4$. Together they imply that $\|\mathcal{P}_{T^\perp}(\boldsymbol{Y})\| < 1$ with probability at least $1 - cn^{-\beta}$ provided that the number of samples obeys

$$m \geq C \max(\mu_0^2 r, \mu_0 n^{1/5}) \, nr\beta \log n \tag{4.20}$$

for some constant C. The two expressions in the maximum come from Lemmas 4.7 and 4.8 in this order. The reason for this simplified formulation is that the terms μ_1^2, $\mu_0^{1/2}\mu_1$ and $\mu_0^{4/3}r^{1/3}$ which come from Lemmas 4.4, 4.5 and 4.6 are bounded above by $\mu_0^2 r$ since $\mu_1 \leq \mu_0\sqrt{r}$. When $\mu_0 r \leq n^{1/5}$, the recovery is exact provided that m obeys (1.10).

5 Connections with Random Graph Theory

5.1 The injectivity property and the coupon collector's problem

We argued in the Introduction that to have any hope of recovering an unknown matrix of rank 1 by any method whatsoever, one needs at least one observation per row and one observation per

column. Sample m entries uniformly at random. Viewing the row indices as bins, assign the kth sampled entry to the bin corresponding to its row index. Then to have any hope of recovering our matrix, all the bins need to be occupied. Quantifying how many samples are required to fill all of the bins is the famous *coupon collector's problem.*

Coupon collection is also connected to the injectivity of the sampling operator $\mathcal{P}_\Omega$ restricted to elements in T. Suppose we sample the entries of a rank 1 matrix equal to $\boldsymbol{x}\boldsymbol{y}^*$ with left and right singular vectors $\boldsymbol{u} = \boldsymbol{x}/\|\boldsymbol{x}\|$ and $\boldsymbol{v} = \boldsymbol{y}/\|\boldsymbol{y}\|$ respectively and have not seen anything in the ith row. Then we claim that $\mathcal{P}_\Omega$ (restricted to T) has a nontrivial null space and thus $\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ is not invertible. Indeed, consider the matrix $\boldsymbol{e}_i\boldsymbol{v}^*$. This matrix is in T and

$$\mathcal{P}_\Omega(\boldsymbol{e}_i\boldsymbol{v}^*) = 0$$

since $\boldsymbol{e}_i\boldsymbol{v}^*$ vanishes outside of the ith row. The same applies to the columns as well. If we have not seen anything in column j, then the rank-1 matrix $\boldsymbol{u}\boldsymbol{e}_j^* \in T$ and $\mathcal{P}_\Omega(\boldsymbol{u}\boldsymbol{e}_j^*) = 0$. In conclusion, the invertibility of $\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ implies a complete collection.

When the entries are sampled uniformly at random, it is well known that one needs on the order of $n \log n$ samples to sample all the rows. What is interesting is that Theorem 4.1 implies that $\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ is invertible—a stronger property—when the number of samples is also on the order of $n \log n$. A particular implication of this discussion is that the logarithmic factors in Theorem 4.1 are unavoidable.

5.2   The injectivity property and the connectivity problem

To recover a matrix of rank 1, one needs much more than at least one observation per row and column. Let R be the set of row indices, $1 \le i \le n$, and C be the set of column indices, $1 \le j \le n$, and consider the bipartite graph connecting vertices $i \in R$ to vertices $j \in C$ if and only if $(i,j) \in \Omega$, i.e. the (i,j)th entry is observed. We claim that if this graph is not fully connected, then one cannot hope to recover a matrix of rank 1.

To see this, we let I be the set of row indices and J be the set of column indices in any connected component. We will assume that I and J are nonempty as otherwise, one is in the previously discussed situation where some rows or columns are not sampled. Consider a rank 1 matrix equal to $\boldsymbol{x}\boldsymbol{y}^*$ as before with singular vectors $\boldsymbol{u} = \boldsymbol{x}/\|\boldsymbol{x}\|$ and $\boldsymbol{v} = \boldsymbol{y}/\|\boldsymbol{y}\|$. Then all the information about the values of the x_i's with $i \in I$ and of the y_j's with $j \in J$ are given by the sampled entries connecting I to J since all the other observed entries connect vertices in I^c to those in J^c. Now even if one observes all the entries x_iy_j with $i \in I$ and $j \in J$, then at least the signs of x_i, $i \in I$, and of y_j, $j \in J$, would remain undetermined. Indeed, if the values $(x_i)_{i\in I}$, $(y_j)_{j\in J}$ are consistent with the observed entries, so are the values $(-x_i)_{i\in I}$, $(-y_j)_{j\in J}$. However, since the same analysis holds for the sets I^c and J^c, there are at least two matrices consistent with the observed entries and exact matrix completion is impossible.

The connectivity of the graph is also related to the injectivity of the sampling operator $\mathcal{P}_\Omega$ restricted to elements in T. If the graph is not fully connected, then we claim that $\mathcal{P}_\Omega$ (restricted to T) has a nontrivial null space and thus $\mathcal{P}_T\mathcal{P}_\Omega\mathcal{P}_T$ is not invertible. Indeed, consider the matrix

$$\boldsymbol{M} = \boldsymbol{a}\boldsymbol{v}^* + \boldsymbol{u}\boldsymbol{b}^*,$$

where $a_i = -u_i$ if $i \in I$ and $a_i = u_i$ otherwise, and $b_j = v_j$ if $j \in J$ and $b_j = -v_j$ otherwise. Then this matrix is in T and obeys

$$M_{ij} = 0$$

if $(i,j) \in I \times J$ or $(i,j) \in I^c \times J^c$. Note that on the complement, i.e. $(i,j) \in I \times J^c$ or $(i,j) \in I^c \times J$, one has $M_{ij} = 2u_i v_j$ and one can show that $\boldsymbol{M} \neq 0$ unless $\boldsymbol{uv}^* = 0$. Since Ω is included in the union of $I \times J$ and $I^c \times J^c$, we have that $\mathcal{P}_\Omega(\boldsymbol{M}) = 0$. In conclusion, the invertibility of $\mathcal{P}_T \mathcal{P}_\Omega \mathcal{P}_T$ implies a fully connected graph.

When the entries are sampled uniformly at random, it is well known that one needs on the order of $n \log n$ samples to obtain a fully connected graph with large probability (see, e.g., [8]). Remarkably, Theorem 4.1 implies that $\mathcal{P}_T \mathcal{P}_\Omega \mathcal{P}_T$ is invertible—a stronger property—when the number of samples is also on the order of $n \log n$.

6   Proofs of the Critical Lemmas

In this section, we prove the five lemmas of Section 4.3. Before we begin, however, we develop a simple estimate which we will use throughout. For each pair (a,b) and (a',b'), it follows from the expression of $\mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*)$ (4.6) that

$$\langle \mathcal{P}_T(\boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle = \langle \boldsymbol{e}_a, \boldsymbol{P}_U \boldsymbol{e}_{a'} \rangle 1_{\{b=b'\}} + \langle \boldsymbol{e}_b, \boldsymbol{P}_V \boldsymbol{e}_{b'} \rangle 1_{\{a=a'\}} - \langle \boldsymbol{e}_a, \boldsymbol{P}_U \boldsymbol{e}_{a'} \rangle \langle \boldsymbol{e}_b, \boldsymbol{P}_V \boldsymbol{e}_{b'} \rangle. \tag{6.1}$$

Fix μ_0 obeying $\mu(U) \leq \mu_0$ and $\mu(V) \leq \mu_0$ and note that

$$|\langle \boldsymbol{e}_a, \boldsymbol{P}_U \boldsymbol{e}_{a'} \rangle| = |\langle \boldsymbol{P}_U \boldsymbol{e}_a, \boldsymbol{P}_U \boldsymbol{e}_{a'} \rangle| \leq \|\boldsymbol{P}_U \boldsymbol{e}_a\| \, \|\boldsymbol{P}_U \boldsymbol{e}_{a'}\| \leq \mu_0 r / n_1$$

and similarly for $\langle \boldsymbol{e}_b, \boldsymbol{P}_V \boldsymbol{e}_{b'} \rangle$. Suppose that $b = b'$ and $a \neq a'$, then

$$|\langle \mathcal{P}_T(\boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle| = |\langle \boldsymbol{e}_a, \boldsymbol{P}_U \boldsymbol{e}_{a'} \rangle|(1 - \|\boldsymbol{P}_V \boldsymbol{e}_b\|^2) \leq \mu_0 r / n_1.$$

We have a similar bound when $a = a'$ and $b \neq b'$ whereas when $a \neq a'$ and $b \neq b'$,

$$|\langle \mathcal{P}_T(\boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle| \leq (\mu_0 r)^2 / (n_1 n_2).$$

In short, it follows from this analysis (and from (4.8) for the case where $(a,b) = (a',b')$) that

$$\max_{ab,a'b'} |\langle \mathcal{P}_T(\boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle| \leq 2\mu_0 r / \min(n_1, n_2). \tag{6.2}$$

A consequence of (4.8) is the estimate:

$$\sum_{a'b'} |\langle \mathcal{P}_T(\boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle|^2 = \sum_{a'b'} |\langle \mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*), \boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^* \rangle|^2$$
$$= \|\mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*)\|_F^2 \leq 2\mu_0 r / \min(n_1, n_2), \tag{6.3}$$

which we will apply several times. A related estimate is this:

$$\max_a \sum_b |E_{ab}|^2 \leq \mu_0 r / \min(n_1, n_2), \tag{6.4}$$

and the same is true by exchanging the role of a and b. To see this, write

$$\sum_b |E_{ab}|^2 = \|\boldsymbol{e}_a^* \boldsymbol{E}\|^2 = \|\sum_{j \leq r} \boldsymbol{v}_j \langle \boldsymbol{u}_j, e_a \rangle\|^2 = \sum_{j \leq r} |\langle \boldsymbol{u}_j, e_a \rangle|^2 = \|\boldsymbol{P}_U e_a\|^2,$$

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and the conclusion follows from the coherence property.

We will prove the lemmas in the case where $n_1 = n_2 = n$ for simplicity, i.e. in the case of square matrices of dimension n. The general case is treated in exactly the same way. In fact, the argument only makes use of the bounds (6.2), (6.3) (and sometimes (6.4)), and the general case is obtained by replacing n with $\min(n_1, n_2)$.

Each of the following subsections computes the operator norm of some random variable. In each section, we denote $\boldsymbol{S}$ as the quantity whose norm we wish to analyze. We will also frequently use the notation $\boldsymbol{H}$ for some auxiliary matrix variable whose norm we will need to bound. Hence, we will reuse the same notation many times rather than introducing a dozens new names—just like in computer programming where one uses the same variable name in distinct routines.

6.1 Proof of Lemma 4.4

In this section, we develop a bound on

$$p^{-1}\|\mathcal{P}_{T^\perp}\mathcal{P}_\Omega\mathcal{P}_T(\boldsymbol{E})\| = p^{-1}\|\mathcal{P}_{T^\perp}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{P}_T(\boldsymbol{E})\|$$
$$\leq p^{-1}\|(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{E})\|,$$

where the equality follows from $\mathcal{P}_{T^\perp}\mathcal{P}_T = 0$, and the inequality from $\mathcal{P}_T(\boldsymbol{E}) = \boldsymbol{E}$ together with $\|\mathcal{P}_{T^\perp}(\boldsymbol{X})\| \leq \|\boldsymbol{X}\|$ which is valid for any matrix $\boldsymbol{X}$. Set

$$\boldsymbol{S} \equiv p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{E}) = p^{-1}\sum_{ab}(\delta_{ab} - p)E_{ab}\boldsymbol{e}_a\boldsymbol{e}_b^*. \tag{6.5}$$

We think of $\boldsymbol{S}$ as a random variable since it depends on the random δ_{ab}'s, and note that $\mathbb{E}\,\boldsymbol{S} = 0$.

The proof of Lemma 4.4 operates by developing an estimate on the size of $(\mathbb{E}\,\|\boldsymbol{S}\|^q)^{1/q}$ for some $q \geq 1$ and by applying Markov inequality to bound the tail of the random variable $\|\boldsymbol{S}\|$. To do this, we shall use a symmetrization argument and the noncommutative Khintchine inequality. Since the function $f(\boldsymbol{S}) = \|\boldsymbol{S}\|^q$ is convex, Jensen's inequality gives that

$$\mathbb{E}\,\|\boldsymbol{S}\|^q \leq \mathbb{E}\,\|\boldsymbol{S} - \boldsymbol{S}'\|^q,$$

where $\boldsymbol{S}' = p^{-1}\sum_{ab}(\delta'_{ab} - p)E_{ab}\boldsymbol{e}_a\boldsymbol{e}_b^*$ is an independent copy of $\boldsymbol{S}$. Since $(\delta_{ab} - \delta'_{ab})$ is symmetric, $\boldsymbol{S} - \boldsymbol{S}'$ has the same distribution as

$$p^{-1}\sum_{ab}\epsilon_{ab}(\delta_{ab} - \delta'_{ab})E_{ab}\boldsymbol{e}_a\boldsymbol{e}_b^* \equiv \boldsymbol{S}_\epsilon - \boldsymbol{S}'_\epsilon,$$

where $\{\epsilon_{ab}\}$ is an independent Rademacher sequence and $\boldsymbol{S}_\epsilon = p^{-1}\sum_{ab}\epsilon_{ab}\delta_{ab}E_{ab}\boldsymbol{e}_a\boldsymbol{e}_b^*$. Further, the triangle inequality gives

$$(\mathbb{E}\,\|\boldsymbol{S}_\epsilon - \boldsymbol{S}'_\epsilon\|^q)^{1/q} \leq (\mathbb{E}\,\|\boldsymbol{S}_\epsilon\|^q)^{1/q} + (\mathbb{E}\,\|\boldsymbol{S}'_\epsilon\|^q)^{1/q} = 2(\mathbb{E}\,\|\boldsymbol{S}_\epsilon\|^q)^{1/q}$$

since $\boldsymbol{S}_\epsilon$ and $\boldsymbol{S}'_\epsilon$ have the same distribution and, therefore,

$$(\mathbb{E}\,\|\boldsymbol{S}\|^q)^{1/q} \leq 2p^{-1}\left(\mathbb{E}_\delta\,\mathbb{E}_\epsilon\,\|\sum_{ab}\epsilon_{ab}\delta_{ab}\,E_{ab}\boldsymbol{e}_a\boldsymbol{e}_b^*\|^q\right)^{1/q}.$$

We are now in position to apply the noncommutative Khintchine inequality which bounds the Schatten norm of a Rademacher series. For $q \geq 1$, the *Schatten q-norm* of a matrix is denoted by

$$\|\boldsymbol{X}\|_{S_q} = \left(\sum_{i=1}^{n} \sigma_i(\boldsymbol{X})^q \right)^{1/q}.$$

Note that the nuclear norm is equal to the Schatten 1-norm and the Frobenius norm is equal to the Schatten 2-norm. The following theorem was originally proven by Lust-Picquard [25], and was later sharpened by Buchholz [9].

Lemma 6.1 (Noncommutative Khintchine inequality) *Let $(\boldsymbol{X}_i)_{1 \leq i \leq r}$ be a finite sequence of matrices of the same dimension and let $\{\epsilon_i\}$ be a Rademacher sequence. For each $q \geq 2$*

$$\left[\mathbb{E}_\epsilon \left\| \sum_i \epsilon_i \boldsymbol{X}_i \right\|_{S_q}^q \right]^{1/q} \leq C_K \sqrt{q} \max \left[\left\| \left(\sum_i \boldsymbol{X}_i^* \boldsymbol{X}_i \right)^{1/2} \right\|_{S_q}, \left\| \left(\sum_i \boldsymbol{X}_i \boldsymbol{X}_i^* \right)^{1/2} \right\|_{S_q} \right],$$

where $C_K = 2^{-1/4}\sqrt{\pi/e}$.

For reference, if $\boldsymbol{X}$ is an $n \times n$ matrix and $q \geq \log n$, we have

$$\|\boldsymbol{X}\| \leq \|\boldsymbol{X}\|_{S_q} \leq e\|\boldsymbol{X}\|,$$

so that the Schatten q-norm is within a multiplicative constant from the operator norm. Observe now that with $q' \geq q$

$$\left(\mathbb{E}_\delta \mathbb{E}_\epsilon \|\boldsymbol{S}_\epsilon\|^q \right)^{1/q} \leq \left(\mathbb{E}_\delta \mathbb{E}_\epsilon \|\boldsymbol{S}_\epsilon\|_{S_{q'}}^q \right)^{1/q} \leq \left(\mathbb{E}_\delta \mathbb{E}_\epsilon \|\boldsymbol{S}_\epsilon\|_{S_{q'}}^{q'} \right)^{1/q'}.$$

We apply the noncommutative Khintchine inequality with $q' \geq \log n$, and after a little algebra, obtain

$$\left(\mathbb{E}_\delta \mathbb{E}_\epsilon \|\boldsymbol{S}_\epsilon\|_{S_{q'}}^{q'} \right)^{1/q'} \leq C_K \frac{e\sqrt{q'}}{p} \left(\mathbb{E}_\delta \max \left[\| \sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_a \boldsymbol{e}_a^* \|^{q'/2}, \| \sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_b \boldsymbol{e}_b^* \|^{q'/2} \right] \right)^{1/q'}.$$

The two terms in the right-hand side are essentially the same and if we can bound any one of them, the same technique will apply to the other. We consider the first and since $\sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_a \boldsymbol{e}_a^*$ is a diagonal matrix,

$$\| \sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_a \boldsymbol{e}_a^* \| = \max_a \sum_b \delta_{ab} E_{ab}^2.$$

The following lemma bounds the qth moment of this quantity.

Lemma 6.2 *Suppose that q is an integer obeying $1 \leq q \leq np$ and assume $np \geq 2\log n$. Then*

$$\mathbb{E}_\delta \left(\max_a \sum_b \delta_{ab} E_{ab}^2 \right)^q \leq 2 \left(2np \|\boldsymbol{E}\|_\infty^2 \right)^q. \tag{6.6}$$

The proof of this lemma is in the Appendix. The same estimate applies to $\mathbb{E} \left(\max_b \sum_a \delta_{ab} E_{ab}^2 \right)^q$ and thus for each $q \geq 1$

$$\mathbb{E}_\delta \max \left[\| \sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_a \boldsymbol{e}_a^* \|^q, \| \sum_{ab} \delta_{ab} E_{ab}^2 \boldsymbol{e}_b \boldsymbol{e}_b^* \|^q \right] \leq 4 \left(2np \, \|\boldsymbol{E}\|_\infty^2 \right)^q.$$

(In the rectangular case, the same estimate holds with $n = \max(n_1, n_2)$.)

Take $q = \beta \log n$ for some $\beta \geq 1$, and set $q' = q$. Then since $\|\boldsymbol{E}\|_\infty \leq \mu_1 \sqrt{r}/n$, we established that

$$(\mathbb{E} \, \|\boldsymbol{S}\|^q)^{1/q} \leq C \, \frac{1}{p} \, \sqrt{\beta \log n} \, \sqrt{np} \, \|\boldsymbol{E}\|_\infty = C \, \mu_1 \, \sqrt{\frac{nr \, \beta \log n}{m}} \equiv K_0.$$

Then by Markov's inequality, for each $t > 0$,

$$\mathbb{P}(\|\boldsymbol{S}\| > tK_0) \leq t^{-q},$$

and for $t = e$, we conclude that

$$\mathbb{P} \left(\|\boldsymbol{S}\| > Ce \, \mu_1 \, \sqrt{\frac{nr \, \beta \log n}{m}} \right) \leq n^{-\beta}$$

with the proviso that $m \geq \max(\beta, 2) \, n \log n$ so that Lemma 6.2 holds.

We have not made any assumption in this section about the matrix $\boldsymbol{E}$ (except that we have a bound on the maximum entry) and, therefore, have proved the theorem below, which shall be used many times in the sequel.

Theorem 6.3 *Let $\boldsymbol{X}$ be a fixed $n \times n$ matrix. There is a constant C_0 such that for each $\beta > 2$*

$$p^{-1} \|(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{X})\| \leq C_0 \left(\frac{\beta n \log n}{p} \right)^{1/2} \|\boldsymbol{X}\|_\infty \tag{6.7}$$

with probability at least $1 - n^{-\beta}$ provided that $np \geq \beta \log n$.

Note that this is the same C_0 described in Lemma 4.4.

6.2   Proof of Lemma 4.5

We now need to bound the spectral norm of $\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T \, \mathcal{H}(\boldsymbol{E})$ and will use some of the ideas developed in the previous section. Just as before,

$$p^{-1} \|\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T \, \mathcal{H}(\boldsymbol{E})\| \leq p^{-1} \|(\mathcal{P}_\Omega - p\mathcal{I}) \, \mathcal{H}(\boldsymbol{E})\|,$$

and put

$$\boldsymbol{S} \equiv p^{-1}(\mathcal{P}_\Omega - p\mathcal{I}) \, \mathcal{H}(E) = p^{-2} \sum_{ab, a'b'} \xi_{ab} \xi_{a'b'} \, E_{a'b'} \, \langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*, \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle \boldsymbol{e}_a \boldsymbol{e}_b^*,$$

where here and below, $\xi_{ab} \equiv \delta_{ab} - p$. Decompose $\boldsymbol{S}$ as

$$\boldsymbol{S} \;=\; p^{-2} \sum_{(a,b)=(a',b')} + \; p^{-2} \sum_{(a,b) \neq (a',b')} \;\equiv\; \boldsymbol{S}_0 + \boldsymbol{S}_1. \tag{6.8}$$

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We bound the spectral norm of the diagonal and off-diagonal contributions separately.

We begin with S_0 and decompose $(\xi_{ab})^2$ as

$$\xi_{ab}^2 = (\delta_{ab} - p)^2 = (1 - 2p)(\delta_{ab} - p) + p(1 - p) = (1 - 2p)\xi_{ab} + p(1 - p),$$

which allows us to express S_0 as

$$S_0 = \frac{1 - 2p}{p} \sum_{ab} \xi_{ab} H_{ab} e_a e_b^* + (1 - p) \sum_{ab} H_{ab} e_a e_b^*, \quad H_{ab} \equiv p^{-1} E_{ab} \langle \mathcal{P}_T e_a e_b^*, e_a e_b^* \rangle. \qquad (6.9)$$

Theorem 6.3 bounds the spectral norm of the first term of the right-hand side and we have

$$p^{-1} \| \sum_{ab} \xi_{ab} H_{ab} e_a e_b^* \| \le C_0 \sqrt{\frac{n^3 \beta \log n}{m}} \, \|H\|_\infty$$

with probability at least $1 - n^{-\beta}$. Now since $\|E\|_\infty \le \mu_1 \sqrt{r}/n$ and $|\langle \mathcal{P}_T e_a e_b^*, e_a e_b^* \rangle| \le 2\mu_0 r/n$ by (6.2), $\|H\|_\infty \le \mu_0 \mu_1 (2r/np) \sqrt{r}/n$, and

$$p^{-1} \| \sum_{ab} \xi_{ab} H_{ab} e_a e_b^* \| \le C \mu_0 \mu_1 \frac{nr}{m} \sqrt{\frac{nr \beta \log n}{m}}$$

with the same probability. The second term of the right-hand side in (6.9) is deterministic and we develop an argument that we will reuse several times. We record a useful lemma.

Lemma 6.4 *Let X be a fixed matrix and set $Z \equiv \sum_{ab} X_{ab} \langle \mathcal{P}_T(e_a e_b^*), e_a e_b^* \rangle e_a e_b^*$. Then*

$$\|Z\| \le \frac{2\mu_0 r}{n} \|X\|.$$

Proof Let Λ_U and Λ_V be the diagonal matrices with entries $\|P_U e_a\|^2$ and $\|P_V e_b\|^2$ respectively,

$$\Lambda_U = \mathrm{diag}(\|P_U e_a\|^2), \quad \Lambda_V = \mathrm{diag}(\|P_V e_b\|^2). \qquad (6.10)$$

To bound the spectral norm of Z, observe that it follows from (4.7) that

$$Z = \Lambda_U X + X \Lambda_V - \Lambda_U X \Lambda_V = \Lambda_U X (I - \Lambda_V) + X \Lambda_V. \qquad (6.11)$$

Hence, since $\|\Lambda_U\|$ and $\|\Lambda_V\|$ are bounded by $\min(\mu_0 r/n, 1)$ and $\|I - \Lambda_V\| \le 1$, we have

$$\|Z\| \le \|\Lambda_U\| \|X\| \|I - \Lambda_V\| + \|X\| \|\Lambda_V\| \le (2\mu_0 r/n) \|X\|.$$

∎

Clearly, this lemma and $\|E\| = 1$ give that H defined in (6.9) obeys $\|H\| \le 2\mu_0 r/np$. In summary,

$$\|S_0\| \le C \frac{nr}{m} \left(\mu_0 \mu_1 \sqrt{\frac{\beta nr \log n}{m}} + \mu_0 \right)$$

for some $C > 0$ with the same probability as in Lemma 4.4.

It remains to bound the off-diagonal term. To this end, we use a useful decoupling lemma:

Lemma 6.5 *[16] Let $\{\eta_i\}_{1 \le i \le n}$ be a sequence of independent random variables, and $\{x_{ij}\}_{i \ne j}$ be elements taken from a Banach space. Then*

$$\mathbb{P}(\|\sum_{i \ne j} \eta_i \eta_j x_{ij}\| \ge t) \le C_D \, \mathbb{P}(\|\sum_{i \ne j} \eta_i \eta'_j x_{ij}\| > t/C_D), \tag{6.12}$$

where $\{\eta'_i\}$ is an independent copy of $\{\eta_i\}$.

This lemma asserts that it is sufficient to estimate $\mathbb{P}(\|\boldsymbol{S}'_1\| \ge t)$ where $\boldsymbol{S}'_1$ is given by

$$\boldsymbol{S}'_1 \equiv p^{-2} \sum_{ab \ne a'b'} \xi_{ab} \xi'_{a'b'} \, E_{a'b'} \langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}^*_{b'}, \boldsymbol{e}_a \boldsymbol{e}^*_b \rangle \boldsymbol{e}_a \boldsymbol{e}^*_b \tag{6.13}$$

in which $\{\xi'_{ab}\}$ is an independent copy of $\{\xi_{ab}\}$. We write $\boldsymbol{S}'_1$ as

$$\boldsymbol{S}'_1 = p^{-1} \sum_{ab} \xi_{ab} \, H_{ab} \boldsymbol{e}_a \boldsymbol{e}^*_b, \qquad H_{ab} \equiv p^{-1} \sum_{a'b':(a',b') \ne (a,b)} \xi'_{a'b'} \, E_{a'b'} \langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}^*_{b'}, \boldsymbol{e}_a \boldsymbol{e}^*_b \rangle. \tag{6.14}$$

To bound the tail of $\|\boldsymbol{S}'_1\|$, observe that

$$\mathbb{P}(\|\boldsymbol{S}'_1\| \ge t) \le \mathbb{P}(\|\boldsymbol{S}'_1\| \ge t \mid \|\boldsymbol{H}\|_\infty \le K) + \mathbb{P}(\|\boldsymbol{H}\|_\infty > K).$$

By independence, the first term of the right-hand side is bounded by Theorem 6.3. On the event $\{\|\boldsymbol{H}\|_\infty \le K\}$, we have

$$p^{-1} \|\sum_{ab} \xi_{ab} \, H_{ab} \boldsymbol{e}_a \boldsymbol{e}^*_b\| \le C \sqrt{\frac{n^3 \beta \log n}{m}} \, K.$$

with probability at least $1 - n^{-\beta}$. To bound $\|\boldsymbol{H}\|_\infty$, we use Bernstein's inequality.

Lemma 6.6 *Let $\boldsymbol{X}$ be a fixed matrix and define $\mathcal{Q}(\boldsymbol{X})$ as the matrix whose (a,b)th entry is*

$$[\mathcal{Q}(\boldsymbol{X})]_{ab} = p^{-1} \sum_{a'b':(a',b') \ne (a,b)} (\delta_{a'b'} - p) \, X_{a'b'} \langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}^*_{b'}, \boldsymbol{e}_a \boldsymbol{e}^*_b \rangle,$$

where $\{\delta_{ab}\}$ is an independent Bernoulli sequence obeying $\mathbb{P}(\delta_{ab} = 1) = p$. Then

$$\mathbb{P}\left(\|\mathcal{Q}(\boldsymbol{X})\|_\infty > \lambda \sqrt{\frac{\mu_0 r}{np}} \|\boldsymbol{X}\|_\infty \right) \le 2n^2 \exp\left(-\frac{\lambda^2}{2 + \frac{2}{3} \sqrt{\frac{\mu_0 r}{np}} \lambda} \right). \tag{6.15}$$

With $\lambda = \sqrt{3\beta \log n}$, the right-hand side is bounded by $2n^{2-\beta}$ provided that $np \ge \frac{4\beta}{3} \mu_0 r \log n$. In particular, for $\lambda = \sqrt{6\beta \log n}$ with $\beta > 2$, the bound is less than $2n^{-\beta}$ provided that $np \ge \frac{8\beta}{3} \mu_0 r \log n$.

Proof The inequality (6.15) is an application of Bernstein's inequality, which states that for a sum of uniformly bounded independent zero-mean random variables obeying $|Y_k| \le c$,

$$\mathbb{P}\left(|\sum_{k=1}^{n} Y_k| > t \right) \le 2e^{-t^2/(2\sigma^2 + 2ct/3)}, \tag{6.16}$$

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where σ^2 is the sum of the variances, $\sigma^2 \equiv \sum_{k=1}^{n} \mathrm{Var}(Y_k)$. We have

$$
\mathrm{Var}([\mathcal{Q}(\boldsymbol{X})]_{ab}) = \frac{1-p}{p} \sum_{a'b':(a',b')\neq(a,b)} |X_{a'b'}|^2 |\langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*, \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle|^2
$$

$$
\leq \frac{1-p}{p} \|\boldsymbol{X}\|_\infty^2 \sum_{a'b':(a',b')\neq(a,b)} |\langle \mathcal{P}_T \boldsymbol{e}_a \boldsymbol{e}_b^*, \boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^* \rangle|^2 \leq \frac{1-p}{p} \|\boldsymbol{X}\|_\infty^2 \, 2\mu_0 r/n
$$

by (6.3). Also,

$$
p^{-1} |(\delta_{a'b'} - p) X_{a'b'} \langle \mathcal{P}_T \boldsymbol{e}_{a'} \boldsymbol{e}_{b'}^*, \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle| \leq p^{-1} \|\boldsymbol{X}\|_\infty \, 2\mu_0 r/n
$$

and hence, for each $t > 0$, (6.16) gives

$$
\mathbb{P}(|[\mathcal{Q}(\boldsymbol{X})]_{ab}| > t) \leq 2 \exp\left(-\frac{t^2}{2\frac{\mu_0 r}{np} \|\boldsymbol{X}\|_\infty^2 + \frac{2}{3}\frac{\mu_0 r}{np} \|\boldsymbol{X}\|_\infty t} \right). \tag{6.17}
$$

Putting $t = \lambda\sqrt{\mu_0 r/np}\|\boldsymbol{X}\|_\infty$ for some $\lambda > 0$ and applying the union bound gives (6.15). ∎

Since $\|\boldsymbol{E}\|_\infty \leq \mu_1\sqrt{r}/n$ it follows that $\boldsymbol{H} = \mathcal{Q}(\boldsymbol{E})$ introduced in (6.14) obeys

$$
\|\boldsymbol{H}\|_\infty \leq C\frac{\mu_1\sqrt{r}}{n}\sqrt{\frac{\mu_0 n r\beta\log n}{m}}
$$

with probability at least $1 - 2n^{-\beta}$ for each $\beta > 2$ and, therefore,

$$
\|\boldsymbol{S}_1'\| \leq C\,\sqrt{\mu_0}\mu_1\frac{nr\beta\log n}{m}
$$

with probability at least $1 - 3n^{-\beta}$. In conclusion, we have

$$
p^{-1}\|(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}(\boldsymbol{E})\| \leq C\frac{nr}{m}\left(\sqrt{\mu_0}\mu_1 \left(\sqrt{\frac{\mu_0 n r\beta\log n}{m}} + \beta\log n \right) + \mu_0 \right) \tag{6.18}
$$

with probability at least $1 - (1 + 3C_D)n^{-\beta}$. A simple algebraic manipulation concludes the proof of Lemma 4.5. Note that we have not made any assumption about the matrix $\boldsymbol{E}$ and, therefore, established the following:

Lemma 6.7 *Let $\boldsymbol{X}$ be a fixed $n \times n$ matrix. There is a constant C_0' such that*

$$
p^{-2}\|\sum_{(a,b)\neq(a',b')} \xi_{ab}\xi_{a'b'} X_{ab}\langle \mathcal{P}_T(\boldsymbol{e}_{a'}\boldsymbol{e}_{b'}^*), \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle \boldsymbol{e}_a\boldsymbol{e}_b^*\| \leq C_0'\frac{\sqrt{\mu_0 r}\,\beta\log n}{p}\|\boldsymbol{X}\|_\infty \tag{6.19}
$$

with probability at least $1 - O(n^{-\beta})$ for all $\beta > 2$ provided that $np \geq 3\mu_0 r\beta\log n$.

6.3 Proof of Lemma 4.6

To prove Lemma 4.6, we need to bound the spectral norm of $p^{-1}\left(\mathcal{P}_\Omega - p\mathcal{I}\right)\mathcal{H}^2(\boldsymbol{E})$, a matrix given by

$$p^{-3}\sum_{a_1 b_1, a_2 b_2, a_3 b_3} \xi_{a_1 b_1}\xi_{a_2 b_2}\xi_{a_3 b_3} E_{a_3 b_3}\langle \mathcal{P}_T \boldsymbol{e}_{a_3}\boldsymbol{e}_{b_3}^*, \boldsymbol{e}_{a_2}\boldsymbol{e}_{b_2}^*\rangle\langle\mathcal{P}_T \boldsymbol{e}_{a_2}\boldsymbol{e}_{b_2}^*, \boldsymbol{e}_{a_1}\boldsymbol{e}_{b_1}^*\rangle\boldsymbol{e}_{a_1}\boldsymbol{e}_{b_1}^*,$$

where $\xi_{ab} = \delta_{ab} - p$ as before. It is convenient to introduce notations to compress this expression. Set $\omega = (a, b)$ (and $\omega_i = (a_i, b_i)$ for $i = 1, 2, 3$), $\boldsymbol{F}_\omega = \boldsymbol{e}_a\boldsymbol{e}_b^*$, and $P_{\omega'\omega} = \langle\mathcal{P}_T \boldsymbol{e}_{a'}\boldsymbol{e}_{b'}^*, \boldsymbol{e}_a\boldsymbol{e}_b^*\rangle$ so that

$$p^{-1}\left(\mathcal{P}_\Omega - p\mathcal{I}\right)\mathcal{H}^2(\boldsymbol{E}) = p^{-3}\sum_{\omega_1, \omega_2, \omega_3}\xi_{\omega_1}\xi_{\omega_2}\xi_{\omega_3}\, E_{\omega_3}P_{\omega_3\omega_2}P_{\omega_2\omega_1}\boldsymbol{F}_{\omega_1}.$$

Partition the sum depending on whether some of the ω_i's are the same or not

$$\frac{1}{p}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^2(\boldsymbol{E}) = \frac{1}{p^3}\left[\sum_{\omega_1=\omega_2=\omega_3} + \sum_{\omega_1\neq\omega_2=\omega_3} + \sum_{\omega_1=\omega_3\neq\omega_2} + \sum_{\omega_1=\omega_2\neq\omega_3} + \sum_{\omega_1\neq\omega_2\neq\omega_3}\right]. \quad (6.20)$$

The meaning should be clear; for instance, the sum $\sum_{\omega_1\neq\omega_2=\omega_3}$ is the sum over the ω's such that $\omega_2 = \omega_3$ and $\omega_1 \neq \omega_2$. Similarly, $\sum_{\omega_1\neq\omega_2\neq\omega_3}$ is the sum over the ω's such that they are all distinct. The idea is now to use a decoupling argument to bound each sum in the right-hand side of (6.20) (except for the first which does not need to be decoupled) and show that all terms are appropriately small in the spectral norm.

We begin with the first term which is equal to

$$\frac{1}{p^3}\sum_\omega(\xi_\omega)^3\, E_\omega P_{\omega\omega}^2\boldsymbol{F}_\omega = \frac{1-3p+3p^2}{p^3}\sum_\omega\xi_\omega\, E_\omega P_{\omega\omega}^2\boldsymbol{F}_\omega + \frac{1-3p+2p^2}{p^2}\sum_\omega E_\omega P_{\omega\omega}^2\boldsymbol{F}_\omega, \quad (6.21)$$

where we have used the identity

$$(\xi_\omega)^3 = (1 - 3p + 3p^2)\xi_\omega + p(1 - 3p + 2p^2).$$

Set $H_\omega = E_\omega(p^{-1}P_{\omega\omega})^2$. For the first term in the right-hand side of (6.21), we need to control $\|\sum_\omega\xi_\omega\, H_\omega\boldsymbol{F}_\omega\|$. This is easily bounded by Theorem 6.3. Indeed, it follows from

$$|H_\omega| \leq \left(\frac{2\mu_0 r}{np}\right)^2\|\boldsymbol{E}\|_\infty$$

that for each $\beta > 0$,

$$p^{-1}\|\sum_\omega\xi_\omega\, H_\omega\,\boldsymbol{F}_\omega\| \leq C\left(\frac{\mu_0 nr}{m}\right)^2\mu_1\sqrt{\frac{nr\beta\log n}{m}} = C\,\mu_0^2\mu_1\sqrt{\beta\log n}\left(\frac{nr}{m}\right)^{5/2}$$

with probably at least $1 - n^{-\beta}$. For the second term in the right-hand side of (6.21), we apply Lemma 6.4 which gives

$$\|\sum_\omega E_\omega P_{\omega\omega}^2\boldsymbol{F}_\omega\| \leq (2\mu_0 r/n)^2$$

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so that $\|\boldsymbol{H}\| \leq (2\mu_0 r/np)^2$. In conclusion, the first term in (6.20) has a spectral norm which is bounded by

$$C \left(\frac{nr}{m}\right)^2 \left(\mu_0^2 \mu_1 \left(\frac{nr\beta \log n}{m}\right)^{1/2} + \mu_0^2\right)$$

with probability at least $1 - n^{-\beta}$.

We now turn our attention to the second term which can be written as

$$p^{-3} \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} (\xi_{\omega_2})^2 E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1} \boldsymbol{F}_{\omega_1} = \frac{1 - 2p}{p^3} \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} \xi_{\omega_2} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1} \boldsymbol{F}_{\omega_1}$$

$$+ \frac{1 - p}{p^2} \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1} \boldsymbol{F}_{\omega_1}.$$

Put $\boldsymbol{S}_1$ for the first term; bounding $\|\boldsymbol{S}_1\|$ is a simple application of Lemma 6.7 with $X_\omega = p^{-1} E_\omega P_{\omega\omega}$, which gives

$$\|\boldsymbol{S}_1\| \leq C \, \mu_0^{3/2} \mu_1 \left(\beta \log n\right) \left(\frac{nr}{m}\right)^2$$

since $\|\boldsymbol{E}\|_\infty \leq \mu_1 \sqrt{r}/n$. For the second term, we need to bound the spectral norm of $\boldsymbol{S}_2$ where

$$\boldsymbol{S}_2 \equiv p^{-1} \sum_{\omega_1} \xi_{\omega_1} H_{\omega_1} \boldsymbol{F}_{\omega_1}, \qquad H_{\omega_1} = p^{-1} \sum_{\omega_2 : \omega_2 \neq \omega_1} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1}.$$

Note that $\boldsymbol{H}$ is deterministic. The lemma below provides an estimate about $\|\boldsymbol{H}\|_\infty$.

Lemma 6.8 *The matrix $\boldsymbol{H}$ obeys*

$$\|\boldsymbol{H}\|_\infty \leq \frac{\mu_0 r}{np} \left(3\|\boldsymbol{E}\|_\infty + 2\frac{\mu_0 r}{n}\right). \tag{6.22}$$

Proof We begin by rewriting $\boldsymbol{H}$ as

$$pH_\omega = \sum_{\omega'} E_{\omega'} P_{\omega'\omega'} P_{\omega'\omega} - E_\omega P_{\omega\omega}^2.$$

Clearly, $|E_\omega P_{\omega\omega}^2| \leq (\mu_0 r/n)^2 \|\boldsymbol{E}\|_\infty$ so that it suffices to bound the first term, which is the ωth entry of the matrix

$$\sum_{\omega, \omega'} E_{\omega'} P_{\omega'\omega'} P_{\omega'\omega} \boldsymbol{F}_\omega = \mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} + \boldsymbol{E} \boldsymbol{\Lambda}_V - \boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V).$$

Now it is immediate to see that $\boldsymbol{\Lambda}_U \boldsymbol{E} \in T$ and likewise for $\boldsymbol{E} \boldsymbol{\Lambda}_V$. Hence,

$$\|\mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} + \boldsymbol{E} \boldsymbol{\Lambda}_V - \boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V)\|_\infty \leq \|\boldsymbol{\Lambda}_U \boldsymbol{E}\|_\infty + \|\boldsymbol{E} \boldsymbol{\Lambda}_V\|_\infty + \|\mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V)\|_\infty$$

$$\leq 2\|\boldsymbol{E}\|_\infty \mu_0 r/n + \|\mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V)\|_\infty.$$

We finally use the crude estimate

$$\|\mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V)\|_\infty \leq \|\mathcal{P}_T(\boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V)\| \leq 2\|\boldsymbol{\Lambda}_U \boldsymbol{E} \boldsymbol{\Lambda}_V\| \leq 2(\mu_0 r/n)^2$$

to complete the proof of the lemma. ∎

As a consequence of this lemma, Theorem 6.3 gives

$$\|\boldsymbol{S}_2\| \le C\,\sqrt{\beta \log n}\,\left(\frac{nr}{m}\right)^{3/2}(\mu_0\mu_1 + \mu_0^2\sqrt{r})$$

with probability at least $1 - n^{-\beta}$. In conclusion, the second term in (6.20) has spectral norm bounded by

$$C\,\sqrt{\beta \log n}\,\left(\frac{nr}{m}\right)^{3/2}\left(\mu_0\mu_1\sqrt{\frac{\mu_0 nr\beta \log n}{m}} + \mu_0\mu_1 + \mu_0^2\sqrt{r}\right)$$

with probability at least $1 - O(n^{-\beta})$.

We now examine the third term which can be written as

$$p^{-3}\sum_{\omega_1 \ne \omega_2}(\xi_{\omega_1})^2\xi_{\omega_2}\,E_{\omega_1}P_{\omega_1\omega_2}P_{\omega_2\omega_1}\boldsymbol{F}_{\omega_1} = \frac{1-2p}{p^3}\sum_{\omega_1 \ne \omega_2}\xi_{\omega_1}\xi_{\omega_2}\,E_{\omega_1}P^2_{\omega_2\omega_1}\boldsymbol{F}_{\omega_1}$$

$$+ \frac{1-p}{p^2}\sum_{\omega_1 \ne \omega_2}\xi_{\omega_2}\,E_{\omega_1}P^2_{\omega_2\omega_1}\boldsymbol{F}_{\omega_1}.$$

We use the decoupling argument once more so that for the first term of the right-hand side, it suffices to estimate the tail of the norm of

$$\boldsymbol{S}_1 \equiv p^{-1}\sum_{\omega_1}\xi^{(1)}_{\omega_1}E_{\omega_1}H_{\omega_1}\boldsymbol{F}_{\omega_1}, \qquad H_{\omega_1} \equiv p^{-2}\sum_{\omega_2:\omega_2 \ne \omega_1}\xi^{(2)}_{\omega_2}P^2_{\omega_2\omega_1},$$

where $\{\xi^{(1)}_\omega\}$ and $\{\xi^{(2)}_\omega\}$ are independent copies of $\{\xi_\omega\}$. It follows from Bernstein's inequality and the estimates

$$|P_{\omega_2\omega_1}| \le 2\mu_0 r/n$$

and

$$\sum_{\omega_2:\omega_2 \ne \omega_1}|P_{\omega_2\omega_1}|^4 \le \max_{\omega_2:\omega_2 \ne \omega_1}|P_{\omega_2\omega_1}|^2\sum_{\omega_2:\omega_2 \ne \omega_1}|P_{\omega_2\omega_1}|^2 \le \left(\frac{2\mu_0 r}{n}\right)^2\frac{2\mu_0 r}{n}$$

that for each $\lambda > 0$,[4]

$$\mathbb{P}\left(|H_{\omega_1}| > \lambda\left(\frac{2\mu_0 r}{np}\right)^{3/2}\right) \le 2\exp\left(-\frac{\lambda^2}{2 + \frac{2}{3}\lambda\left(\frac{2\mu_0 r}{np}\right)^{1/2}}\right).$$

It is now not hard to see that this inequality implies that

$$P\left(\|\boldsymbol{H}\|_\infty > \sqrt{8\beta \log n}\,\left(\frac{2\mu_0 nr}{m}\right)^{3/2}\right) \le 2\,n^{-2\beta+2}$$

provided that $m \ge \frac{16}{9}\mu_0 nr\,\beta \log n$. As a consequence, for each $\beta > 2$, Theorem 6.3 gives

$$\|\boldsymbol{S}_1\| \le C\,\mu_0^{3/2}\mu_1\,\beta \log n\left(\frac{nr}{m}\right)^2$$

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[4]We would like to remark that one can often get better estimates; when $\omega_1 \ne \omega_2$, the bound $|P_{\omega_2\omega_1}| \le 2\mu_0 r/n$ may be rather crude. Indeed, one can derive better estimates for the random orthogonal model, for example.

with probability at least $1 - 3n^{-\beta}$. The other term is equal to $(1-p)$ times $\sum_{\omega_1} E_{\omega_1} H_{\omega_1} \boldsymbol{F}_{\omega_1}$, and

$$\|\sum_{\omega_1} E_{\omega_1} H_{\omega_1} \boldsymbol{F}_{\omega_1}\| \le \|\sum_{\omega_1} E_{\omega_1} H_{\omega_1} \boldsymbol{F}_{\omega_1}\|_F$$

$$\le \|\boldsymbol{H}\|_\infty \|\boldsymbol{E}\|_F \le C \sqrt{\beta \log n} \left(\frac{\mu_0 nr}{m}\right)^{3/2} \sqrt{r}.$$

In conclusion, the third term in (6.20) has spectral norm bounded by

$$C \mu_0 \sqrt{\beta \log n} \left(\frac{nr}{m}\right)^{3/2} \left(\mu_1 \sqrt{\frac{\mu_0 nr \beta \log n}{m}} + \sqrt{\mu_0 r}\right)$$

with probability at least $1 - O(n^{-\beta})$.

We proceed to the fourth term which can be written as

$$p^{-3} \sum_{\omega_1 \neq \omega_3} (\xi_{\omega_1})^2 \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1} P_{\omega_1 \omega_1} \boldsymbol{F}_{\omega_1} = \frac{1-2p}{p^3} \sum_{\omega_1 \neq \omega_3} \xi_{\omega_1} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1} P_{\omega_1 \omega_1} \boldsymbol{F}_{\omega_1}$$

$$+ \frac{1-p}{p^2} \sum_{\omega_1 \neq \omega_3} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1} P_{\omega_1 \omega_1} \boldsymbol{F}_{\omega_1}.$$

Let $\boldsymbol{S}_1$ be the first term and set $H_{\omega_1} = p^{-2} \sum_{\omega_1 \neq \omega_3} \xi_{\omega_1} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1} \boldsymbol{F}_{\omega_1}$. Then Lemma 6.4 gives

$$\|\boldsymbol{S}_1\| \le \frac{2\mu_0 r}{np} \|\boldsymbol{H}\| \le C \mu_0^{3/2} \mu_1 (\beta \log n) \left(\frac{nr}{m}\right)^2$$

where the last inequality is given by Lemma 6.7. For the other term—call it $\boldsymbol{S}_2$—set $H_{\omega_1} = p^{-1} \sum_{\omega_3 : \omega_3 \neq \omega_1} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1}$. Then Lemma 6.4 gives

$$\|\boldsymbol{S}_2\| \le \frac{2\mu_0 r}{np} \|\boldsymbol{H}\|.$$

Notice that $H_{\omega_1} = p^{-1} \sum_{\omega_3} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_1} - p^{-1} \xi_{\omega_1} E_{\omega_1} P_{\omega_1 \omega_1}$ so that with $G_{\omega_1} = E_{\omega_1} P_{\omega_1 \omega_1}$

$$\boldsymbol{H} = p^{-1} [\mathcal{P}_T (\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{E}) - (\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{G})].$$

Now for any matrix $\boldsymbol{X}$, $\|\mathcal{P}_T(\boldsymbol{X})\| = \|\boldsymbol{X} - \mathcal{P}_{T^\perp}(\boldsymbol{X})\| \le 2\|\boldsymbol{X}\|$ and, therefore,

$$\|\boldsymbol{H}\| \le 2p^{-1} \|(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{E})\| + p^{-1} \|(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{G})\|.$$

As a consequence and since $\|\boldsymbol{G}\|_\infty \le \|\boldsymbol{E}\|_\infty$, Theorem 6.3 gives for each $\beta > 2$,

$$\|\boldsymbol{H}\| \le C\mu_1 \sqrt{\frac{nr \beta \log n}{m}}$$

with probability at least $1 - n^{-\beta}$. In conclusion, the fourth term in (6.20) has spectral norm bounded by

$$C \mu_0 \mu_1 \sqrt{\beta \log n} \left(\frac{nr}{m}\right)^{3/2} \left(\sqrt{\frac{\mu_0 nr \beta \log n}{m}} + 1\right)$$

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with probability at least $1 - O(n^{-\beta})$.

We finally examine the last term

$$p^{-3} \sum_{\omega_1 \neq \omega_2 \neq \omega_3} \xi_{\omega_1} \xi_{\omega_2} \xi_{\omega_3} \, E_{\omega_3} P_{\omega_3 \omega_2} P_{\omega_2 \omega_1} \boldsymbol{F}_{\omega_1}.$$

Now just as one has a decoupling inequality for pairs of variables, we have a decoupling inequality for triples as well and we thus simply need to bound the tail of

$$\boldsymbol{S}_1 \equiv p^{-3} \sum_{\omega_1 \neq \omega_2 \neq \omega_3} \xi_{\omega_1}^{(1)} \xi_{\omega_2}^{(2)} \xi_{\omega_3}^{(3)} \, E_{\omega_3} P_{\omega_3 \omega_2} P_{\omega_2 \omega_1} \boldsymbol{F}_{\omega_1}$$

in which the sequences $\{\xi_\omega^{(1)}\}$, $\{\xi_\omega^{(2)}\}$ and $\{\xi_\omega^{(3)}\}$ are independent copies of $\{\xi_\omega\}$. We refer to [16] for details. We now argue as in Section 6.2 and write $\boldsymbol{S}_1$ as

$$\boldsymbol{S}_1 = p^{-1} \sum_{\omega_1} \xi_{\omega_1}^{(1)} H_{\omega_1} \boldsymbol{F}_{\omega_1},$$

where

$$H_{\omega_1} \equiv p^{-1} \sum_{\omega_2 : \omega_2 \neq \omega_1} \xi_{\omega_2}^{(2)} G_{\omega_2} P_{\omega_2 \omega_1}, \qquad G_{\omega_2} \equiv p^{-1} \sum_{\omega_3 : \omega_3 \neq \omega_1, \omega_3 \neq \omega_2} \xi_{\omega_3}^{(3)} E_{\omega_3} P_{\omega_3 \omega_2}. \qquad (6.23)$$

By Lemma 6.6, we have for each $\beta > 2$

$$\|\boldsymbol{G}\|_\infty \leq C \sqrt{\frac{\mu_0 n r \beta \log n}{m}} \, \|\boldsymbol{E}\|_\infty$$

with large probability and the same argument then gives

$$\|\boldsymbol{H}\|_\infty \leq C \sqrt{\frac{\mu_0 n r \beta \log n}{m}} \, \|\boldsymbol{G}\|_\infty \leq C \frac{\mu_0 n r \beta \log n}{m} \, \|\boldsymbol{E}\|_\infty$$

with probability at least $1 - 4n^{-\beta}$. As a consequence, Theorem 6.3 gives

$$\|\boldsymbol{S}\| \leq C \, \mu_0 \mu_1 \left(\frac{n r \beta \log n}{m} \right)^{3/2}$$

with probability at least $1 - O(n^{-\beta})$.

To summarize the calculations of this section and using the fact that $\mu_0 \geq 1$ and $\mu_1 \leq \mu_0 \sqrt{r}$, we have established that if $m \geq \mu_0 \, nr(\beta \log n)$,

$$p^{-1} \|(\mathcal{P}_\Omega - p\mathcal{I}) \, \mathcal{H}^2(\boldsymbol{E})\| \leq C \left(\frac{nr}{m} \right)^2 \left(\mu_0^2 \mu_1 \sqrt{\frac{nr \beta \log n}{m}} + \mu_0^2 \right)$$

$$+ C \sqrt{\beta \log n} \left(\frac{nr}{m} \right)^{3/2} \mu_0^2 \sqrt{r} + C \left(\frac{nr \beta \log n}{m} \right)^{3/2} \mu_0 \mu_1$$

with probability at least $1 - O(n^{-\beta})$. One can check that if $m = \lambda \mu_0^{4/3} n r^{4/3} \beta \log n$ for a fixed $\beta \geq 2$ and $\lambda \geq 1$, then there is a constant C such that

$$\|p^{-1} (\mathcal{P}_\Omega - p\mathcal{I}) \, \mathcal{H}^2(\boldsymbol{E})\| \leq C \lambda^{-3/2}$$

with probability at least $1 - O(n^{-\beta})$. This is the content of Lemma 4.6.

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6.4 Proof of Lemma 4.7

Clearly, one could continue on the same path and estimate the spectral norm of $p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^3(\boldsymbol{E})$ by the same technique as in the previous sections. That is to say, we would write

$$p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^3(\boldsymbol{E}) = p^{-4} \sum_{\omega_1,\omega_2,\omega_3,\omega_4} \left[\prod_{i=1}^{4} \xi_{\omega_i}\right] E_{\omega_4} \left[\prod_{i=1}^{3} P_{\omega_{i+1}\omega_i}\right] \boldsymbol{F}_{\omega_1}$$

with the same notations as before, and partition the sum depending on whether some of the ω_i's are the same or not. Then we would use the decoupling argument to bound each term in the sum. Although this is a clear possibility, one would need to consider 18 cases and the calculations would become a little laborious. In this section, we propose to bound the term $p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^3(\boldsymbol{E})$ with a different argument which has two main advantages: first, it is much shorter and second, it uses much of what we have already established. The downside is that it is not as sharp.

The starting point is to note that

$$p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^3(\boldsymbol{E}) = p^{-1}(\Xi \circ \mathcal{H}^3(\boldsymbol{E})),$$

where Ξ is the matrix with i.i.d. entries equal to $\xi_{ab} = \delta_{ab} - p$ and $\circ$ denotes the Hadamard product (componentwise multiplication). To bound the spectral norm of this Hadamard product, we apply an inequality due to Ando, Horn, and Johnson [4]. An elementary proof can be found in §5.6 of [19].

Lemma 6.9 *[19] Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n_1 \times n_2$ matrices. Then*

$$\|\boldsymbol{A} \circ \boldsymbol{B}\| \leq \|\boldsymbol{A}\| \, \nu(\boldsymbol{B}), \tag{6.24}$$

where ν is the function
$$\nu(\boldsymbol{B}) = \inf\{c(\boldsymbol{X})c(\boldsymbol{Y}) : \boldsymbol{X}\boldsymbol{Y}^* = \boldsymbol{B}\},$$

and $c(\boldsymbol{X})$ is the maximum Euclidean norm of the rows

$$c(\boldsymbol{X})^2 = \max_{1 \leq i \leq n} \sum_j X_{ij}^2.$$

To apply (6.24), we first notice that one can estimate the norm of Ξ via Theorem 6.3. Indeed, let $\boldsymbol{Z} = \boldsymbol{1}\boldsymbol{1}^*$ be the matrix with all entries equal to one. Then $p^{-1}\Xi = p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})(\boldsymbol{Z})$ and thus

$$p^{-1}\|\Xi\| \leq C \left(\frac{n^3 \beta \log n}{m}\right)^{1/2} \tag{6.25}$$

with probability at least $1 - n^{-\beta}$. One could obtain a similar result by appealing to the recent literature on random matrix theory and on concentration of measure. Potentially this could allow to derive an upper bound without the logarithmic term but we will not consider these refinements here. (It is interesting to note in passing, however, that the two page proof of Theorem 6.3 gives a large deviation result about the largest singular value of a matrix with i.i.d. entries which is sharp up to a multiplicative factor proportional to at most $\sqrt{\log n}$.)

Second, we bound the second factor in (6.24) via the following estimate:

Lemma 6.10 *There are numerical constants C and c so that for each $\beta > 2$, $\mathcal{H}^3(\boldsymbol{E})$ obeys*

$$\nu(\mathcal{H}^3(\boldsymbol{E})) \leq C\mu_0 r/n \tag{6.26}$$

with probability at least $1 - O(n^{-\beta})$ provided that $m \geq c\,\mu_0^{4/3}\,nr^{5/3}(\beta \log n)$.

The two inequalities (6.25) and (6.26) give

$$p^{-1}\|\Xi \circ \mathcal{H}^3(\boldsymbol{E})\| \leq C\sqrt{\frac{\mu_0^2\,nr^2\,\beta \log n}{m}},$$

with large probability. Hence, when m is substantially larger than a constant times $\mu_0^2 nr^2(\beta \log n)$, we have that the spectral norm of $p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\,\mathcal{H}^3(\boldsymbol{E})$ is much less than 1. This is the content of Lemma 4.7.

The remainder of this section proves Lemma 6.10. Set $\boldsymbol{S} \equiv \mathcal{H}^3(\boldsymbol{E})$ for short. Because $\boldsymbol{S}$ is in T, $\boldsymbol{S} = \mathcal{P}_T(\boldsymbol{S}) = \boldsymbol{P}_U\boldsymbol{S} + \boldsymbol{S}\boldsymbol{P}_V - \boldsymbol{P}_U\boldsymbol{S}\boldsymbol{P}_V$. Writing $\boldsymbol{P}_U = \sum_{j=1}^r \boldsymbol{u}_j\boldsymbol{u}_j^*$ and similarly for $\boldsymbol{P}_V$ gives

$$\boldsymbol{S} = \sum_{j=1}^r \boldsymbol{u}_j(\boldsymbol{u}_j^*\boldsymbol{S}) + \sum_{j=1}^r ((I - \boldsymbol{P}_U)\boldsymbol{S}\boldsymbol{v}_j)\boldsymbol{v}_j^*.$$

For each $1 \leq j \leq r$, let $\boldsymbol{\alpha}_j \equiv \boldsymbol{S}\boldsymbol{v}_j$ and $\boldsymbol{\beta}_j^* \equiv \boldsymbol{u}_j^*\boldsymbol{S}$. Then the decomposition

$$\boldsymbol{S} = \sum_{j=1}^r \boldsymbol{u}_j\boldsymbol{\beta}_j^* + \sum_{j=1}^r (\boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_j)\boldsymbol{v}_j^*,$$

where $\boldsymbol{P}_{U^\perp} = I - \boldsymbol{P}_U$, provides a factorization of the form

$$\boldsymbol{S} = \boldsymbol{X}\boldsymbol{Y}^*, \quad \begin{cases} \boldsymbol{X} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r, \boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_1, \ldots, \boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_r], \\ \boldsymbol{Y} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r]. \end{cases}$$

It follows from our assumption that

$$c^2([\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r]) = \max_{1 \leq i \leq n} \sum_{1 \leq j \leq r} u_{ij}^2 = \max_{1 \leq i \leq n} \|\boldsymbol{P}_U\boldsymbol{e}_i\|^2 \leq \mu_0 r/n,$$

and similarly for $[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r]$. Hence, to prove Lemma 6.10, it suffices to prove that the maximum row norm obeys $c([\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r]) \leq C\sqrt{\mu_0 r/n}$ for some constant $C > 0$, and similarly for the matrix $[\boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_1, \ldots, \boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_r]$.

Lemma 6.11 *There is a numerical constant C such that for each $\beta > 2$,*

$$c([\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r]) \leq C\sqrt{\mu_0 r/n} \tag{6.27}$$

with probability at least $1 - O(n^{-\beta})$ provided that m obeys the condition of Lemma 6.10.

A similar estimate for $[\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r]$ is obtained in the same way by exchanging the roles of $\boldsymbol{u}$ and $\boldsymbol{v}$. Moreover, a minor modification of the argument gives

$$c([\boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_1, \ldots, \boldsymbol{P}_{U^\perp}\boldsymbol{\alpha}_r]) \leq C\sqrt{\mu_0 r/n} \tag{6.28}$$

as well, and we will omit the details. In short, the estimate (6.27) implies Lemma 6.10.

Proof [of Lemma 6.11] To prove (6.27), we use the notations of the previous section and write

$$\boldsymbol{\alpha}_j = p^{-3} \sum_{a_1 b_1, a_2 b_2, a_3 b_3} \xi_{a_1 b_1} \xi_{a_2 b_2} \xi_{a_3 b_3} E_{a_3 b_3} \langle \mathcal{P}_T \boldsymbol{e}_{a_3} \boldsymbol{e}_{b_3}^*, \boldsymbol{e}_{a_2} \boldsymbol{e}_{b_2}^* \rangle \langle \mathcal{P}_T \boldsymbol{e}_{a_2} \boldsymbol{e}_{b_2}^*, \boldsymbol{e}_{a_1} \boldsymbol{e}_{b_1}^* \rangle \mathcal{P}_T (\boldsymbol{e}_{a_1} \boldsymbol{e}_{b_1}^*) \boldsymbol{v}_j$$

$$= p^{-3} \sum_{\omega_1, \omega_2, \omega_3} \xi_{\omega_1} \xi_{\omega_2} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_2} P_{\omega_2 \omega_1} \mathcal{P}_T (\boldsymbol{F}_{\boldsymbol{\omega}_1}) \boldsymbol{v}_j$$

$$= p^{-3} \sum_{\omega_1, \omega_2, \omega_3} \xi_{\omega_1} \xi_{\omega_2} \xi_{\omega_3} E_{\omega_3} P_{\omega_3 \omega_2} P_{\omega_2 \omega_1} (\boldsymbol{F}_{\boldsymbol{\omega}_1} \boldsymbol{v}_j)$$

since for any matrix $\boldsymbol{X}$, $\mathcal{P}_T(\boldsymbol{X})\boldsymbol{v}_j = \boldsymbol{X}\boldsymbol{v}_j$ for each $1 \le j \le r$. We then follow the same steps as in Section 6.3 and partition the sum depending on whether some of the ω_i's are the same or not

$$\boldsymbol{\alpha}_j = p^{-3} \left[\sum_{\omega_1 = \omega_2 = \omega_3} + \sum_{\omega_1 \ne \omega_2 = \omega_3} + \sum_{\omega_1 = \omega_3 \ne \omega_2} + \sum_{\omega_1 = \omega_2 \ne \omega_3} + \sum_{\omega_1 \ne \omega_2 \ne \omega_3} \right]. \tag{6.29}$$

The idea is this: to establish (6.27), it is sufficient to show that if $\boldsymbol{\gamma}_j$ is any of the five terms above, it obeys

$$\sqrt{\sum_{1 \le j \le r} |\gamma_{ij}|^2} \le C \sqrt{\mu_0 r / n} \tag{6.30}$$

(γ_{ij} is the ith component of $\boldsymbol{\gamma}_j$ as usual) with large probability. The strategy for getting such estimates is to use decoupling whenever applicable.

Just as Theorem 6.3 proved useful to bound the norm of $p^{-1}(\mathcal{P}_\Omega - p\mathcal{I})\mathcal{H}^2(\boldsymbol{E})$ in Section 6.3, the lemma below will help bounding the magnitudes of the components of $\boldsymbol{\alpha}_j$.

Lemma 6.12 *Define* $\boldsymbol{S} \equiv p^{-1} \sum_{ij} \sum_\omega \xi_\omega H_\omega \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle \boldsymbol{e}_i \boldsymbol{e}_j^*$. *Then for each* $\lambda > 0$

$$\mathbb{P}(\|\boldsymbol{S}\|_\infty \ge \sqrt{\mu_0 / n}) \le 2n^2 \exp\left(-\frac{1}{\frac{2n}{\mu_0 p} \|H\|_\infty^2 + \frac{2}{3p} \sqrt{r} \|H\|_\infty} \right). \tag{6.31}$$

Proof The proof is an application of Bernstein's inequality (6.16). Note that $\langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle = 1_{\{a=i\}} v_{bj}$ and hence

$$\mathrm{Var}(S_{ij}) \le p^{-1} \|\boldsymbol{H}\|_\infty^2 \sum_\omega |\langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle|^2 = p^{-1} \|\boldsymbol{H}\|_\infty^2$$

since $\sum_\omega |\langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle|^2 = 1$, and $|p^{-1} H_\omega \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle| \le p^{-1} \|\boldsymbol{H}\|_\infty \sqrt{\mu_0 r / n}$ since $|\langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle| \le |v_{bj}|$ and

$$|v_{bj}| \le \|\boldsymbol{P}_V \boldsymbol{e}_b\| \le \sqrt{\mu_0 r / n}.$$

■

Each term in (6.29) is given by the corresponding term in (6.20) after formally substituting $\boldsymbol{F}_\omega$ with $\boldsymbol{F}_\omega \boldsymbol{v}_j$. We begin with the first term whose ith component is equal to

$$\gamma_{ij} \equiv p^{-3}(1 - 3p + 3p^2) \sum_\omega \xi_\omega E_\omega P_{\omega\omega}^2 \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle + p^{-2}(1 - 3p + 2p^2) \sum_\omega E_\omega P_{\omega\omega}^2 \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle. \tag{6.32}$$

Ignoring the constant factor $(1 - 3p + 3p^2)$ which is bounded by 1, we write the first of these two terms as

$$(\boldsymbol{S}_0)_{ij} \equiv p^{-1} \sum_\omega \xi_\omega \, H_\omega \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle, \qquad H_\omega = E_\omega \, (p^{-1} P_{\omega\omega})^2.$$

Since $\|\boldsymbol{H}\|_\infty \le (\mu_0 nr/m)^2 \, \mu_1 \sqrt{r}/n$, it follows from Lemma (6.12) that

$$\mathbb{P}\left(\|\boldsymbol{S}_0\|_\infty \ge \sqrt{\mu_0/n}\right) \le 2n^2 \, \mathrm{e}^{-1/D}, \quad D \le C\left(\mu_0^3 \mu_1^2 \left(\frac{nr}{m}\right)^5 + \mu_0^2 \mu_1 \left(\frac{nr}{m}\right)^3\right)$$

for some numerical $C > 0$. Since $\mu_1 \le \mu_0 \sqrt{r}$, we have that when $m \ge \lambda\mu_0 \, nr^{6/5}(\beta \log n)$ for some numerical constant $\lambda > 0$, $\|\boldsymbol{S}_0\|_\infty \ge \sqrt{\mu_0/n}$ with probability at most $2n^2 e^{-(\beta \log n)^3}$; this probability is inversely proportional to a superpolynomial in n. For the second term, the matrix with entries $E_\omega P_{\omega\omega}^2$ is given by

$$\boldsymbol{\Lambda}_U^2 \boldsymbol{E} + \boldsymbol{E}\boldsymbol{\Lambda}_V^2 + 2\boldsymbol{\Lambda}_U \boldsymbol{E}\boldsymbol{\Lambda}_V + \boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{\Lambda}_V^2 - 2\boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{\Lambda}_V - 2\boldsymbol{\Lambda}_U \boldsymbol{E}\boldsymbol{\Lambda}_V^2$$

and thus

$$\sum_\omega E_\omega P_{\omega\omega}^2 \langle \boldsymbol{e}_i, \boldsymbol{F}_\omega \boldsymbol{v}_j \rangle = \langle \boldsymbol{e}_i, (\boldsymbol{\Lambda}_U^2 \boldsymbol{E} + \boldsymbol{E}\boldsymbol{\Lambda}_V^2 + 2\boldsymbol{\Lambda}_U \boldsymbol{E}\boldsymbol{\Lambda}_V + \boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{\Lambda}_V^2 - 2\boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{\Lambda}_V - 2\boldsymbol{\Lambda}_U \boldsymbol{E}\boldsymbol{\Lambda}_V^2)\boldsymbol{v}_j \rangle.$$

This is a sum of six terms and we will show how to bound the first three; the last three are dealt in exactly the same way and obey better estimates. For the first, we have

$$\langle \boldsymbol{e}_i, \boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{v}_j \rangle = \langle \boldsymbol{\Lambda}_U^2 \boldsymbol{e}_i, \boldsymbol{E}\boldsymbol{v}_j \rangle = \|\boldsymbol{P}_U \boldsymbol{e}_i\|^4 \langle \boldsymbol{e}_i, \boldsymbol{u}_j \rangle$$

Hence

$$p^{-2}\sqrt{\sum_{1 \le j \le r} |\langle \boldsymbol{e}_i, \boldsymbol{\Lambda}_U^2 \boldsymbol{E}\boldsymbol{v}_j \rangle|^2} = p^{-2}\|\boldsymbol{P}_U \boldsymbol{e}_i\|^4 \sqrt{\sum_{1 \le j \le r} |\langle \boldsymbol{e}_i, \boldsymbol{u}_j \rangle|^2} = p^{-2}\|\boldsymbol{P}_U \boldsymbol{e}_i\|^5 \le \left(\frac{\mu_0 r}{np}\right)^2 \sqrt{\frac{\mu_0 r}{n}}.$$

In other words, when $m \ge \mu_0 nr$, the right hand-side is bounded by $\sqrt{\mu_0 r/n}$ as desired. For the second term, we have

$$\langle \boldsymbol{e}_i, \boldsymbol{E}\boldsymbol{\Lambda}_V^2 \boldsymbol{v}_j \rangle = \sum_b \|\boldsymbol{P}_V \boldsymbol{e}_b\|^4 v_{bj} \langle \boldsymbol{e}_i, \boldsymbol{E}\boldsymbol{e}_b \rangle = \sum_b \|\boldsymbol{P}_V \boldsymbol{e}_b\|^4 v_{bj} E_{ib}.$$

Hence it follows from the Cauchy-Schwarz inequality and (6.4) that

$$p^{-2}|\langle \boldsymbol{e}_i, \boldsymbol{E}\boldsymbol{\Lambda}_V^2 \boldsymbol{v}_j \rangle| \le \left(\frac{\mu_0 r}{np}\right)^2 \sqrt{\frac{\mu_0 r}{n}}.$$

In other words, when $m \ge \mu_0 nr^{5/4}$,

$$p^{-2}\sqrt{\sum_{1 \le j \le r} |\langle \boldsymbol{e}_i, \boldsymbol{E}\boldsymbol{\Lambda}_V^2 \boldsymbol{v}_j \rangle|^2} \le \sqrt{\frac{\mu_0 r}{n}} \tag{6.33}$$

as desired. For the third term, we have

$$\langle \boldsymbol{e}_i, \boldsymbol{\Lambda}_U \boldsymbol{E}\boldsymbol{\Lambda}_V \boldsymbol{v}_j \rangle = \|\boldsymbol{P}_U \boldsymbol{e}_i\|^2 \sum_b \|\boldsymbol{P}_V \boldsymbol{e}_b\|^2 v_{bj} E_{ib}.$$

The Cauchy-Schwarz inequality gives

$$2p^{-2}|\langle e_i, \mathbf{\Lambda}_U \mathbf{E} \mathbf{\Lambda}_V v_j \rangle| \leq 2 \left(\frac{\mu_0 r}{np} \right)^2 \sqrt{\frac{\mu_0 r}{n}}$$

just as before. In other words, when $m \geq \mu_0 n r^{5/4}$, $2p^{-2}\sqrt{\sum_{1 \leq j \leq r} |\langle e_i, \mathbf{\Lambda}_U \mathbf{E} \mathbf{\Lambda}_V v_j \rangle|^2}$ is bounded by $2\sqrt{\mu_0 r/n}$. The other terms obey (6.33) as well when $m \geq \mu_0 n r^{5/4}$. In conclusion, the first term (6.32) in (6.29) obeys (6.30) with probability at least $1 - O(n^{-\beta})$ provided that $m \geq \mu_0 n r^{5/4}(\beta \log n)$.

We now turn our attention to the second term which can be written as

$$\gamma_{ij} \equiv p^{-3}(1-2p) \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} \xi_{\omega_2} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1} \langle e_i, \mathbf{F}_{\omega_1} v_j \rangle$$

$$+ p^{-2}(1-p) \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1} \langle e_i, \mathbf{F}_{\omega_1} v_j \rangle.$$

We decouple the first term so that it suffices to bound

$$(\mathbf{S}_0)_{ij} \equiv p^{-1} \sum_{\omega_1} \xi_{\omega_1}^{(1)} H_{\omega_1} \langle e_i, \mathbf{F}_{\omega_1} v_j \rangle, \qquad H_{\omega_1} \equiv p^{-2} \sum_{\omega_2 : \omega_2 \neq \omega_1} \xi_{\omega_2}^{(2)} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1},$$

where the sequences $\{\xi_\omega^{(1)}\}$ and $\{\xi_\omega^{(2)}\}$ are independent. The method from Section 6.2 shows that

$$\|\mathbf{H}\|_\infty \leq C \sqrt{\frac{\mu_0 n r \beta \log n}{m}} \sup_\omega |E_\omega(p^{-1} P_{\omega\omega})| \leq C \sqrt{\beta \log n} \left(\frac{\mu_0 n r}{m} \right)^{3/2} \|\mathbf{E}\|_\infty$$

with probability at least $1 - 2n^{-\beta}$ for each $\beta > 2$. Therefore, Lemma 6.12 gives

$$\mathbb{P}\left(\|\mathbf{S}_0\|_\infty \geq \sqrt{\mu_0/n} \right) \leq 2n^2 e^{-1/D}, \tag{6.34}$$

where D obeys

$$D \leq C \left(\mu_0^2 \mu_1^2 (\beta \log n) \left(\frac{nr}{m} \right)^4 + \mu_0^{3/2} \mu_1 \sqrt{\beta \log n} \left(\frac{nr}{m} \right)^{5/2} \right). \tag{6.35}$$

for some positive constant C. Hence, when $m \geq \lambda \mu_0 n r^{5/4}(\beta \log n)$ for some sufficiently large numerical constant $\lambda > 0$, we have that $\|\mathbf{S}_0\|_\infty \geq \sqrt{\mu_0/n}$ with probability at most $2n^2 e^{-(\beta \log n)^2}$. This is inversely proportional to a superpolynomial in n. We write the second term as

$$(\mathbf{S}_1)_{ij} \equiv p^{-1} \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} H_{\omega_1} \langle e_i, \mathbf{F}_{\omega_1} v_j \rangle, \qquad H_{\omega_1} = p^{-1} \sum_{\omega_2 : \omega_2 \neq \omega_1} E_{\omega_2} P_{\omega_2 \omega_2} P_{\omega_2 \omega_1}.$$

We know from Section 6.3 that $\mathbf{H}$ obeys $\|\mathbf{H}\|_\infty \leq C \mu_0^2 r^2/m$ since $\mu_1 \leq \mu_0 \sqrt{r}$ so that Lemma 6.12 gives

$$\mathbb{P}\left(\|\mathbf{S}_1\|_\infty \geq \sqrt{\mu_0/n} \right) \leq 2n^2 e^{-1/D}, \quad D \leq C \left(\mu_0^3 \frac{n^3 r^4}{m^3} + \mu_0^2 \frac{n^2 r^{5/2}}{m^2} \right)$$

for some $C > 0$. Hence, when $m \geq \lambda \mu_0 n r^{4/3}(\beta \log n)$ for some numerical constant $\lambda > 0$, we have that $\|\mathbf{S}_1\|_\infty \geq \sqrt{\mu_0/n}$ with probability at most $2n^2 e^{-(\beta \log n)^2}$. This is inversely proportional to a superpolynomial in n. In conclusion and taking into account the decoupling constants in (6.12),

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the second term in (6.29) obeys (6.30) with probability at least $1 - O(n^{-\beta})$ provided that m is sufficiently large as above.

We now examine the third term which can be written as

$$p^{-3}(1-2p) \sum_{\omega_1 \neq \omega_2} \xi_{\omega_1} \xi_{\omega_2} E_{\omega_1} P_{\omega_2\omega_1}^2 \langle e_i, F_{\omega_1} v_j \rangle + p^{-2}(1-p) \sum_{\omega_1 \neq \omega_2} \xi_{\omega_2} E_{\omega_1} P_{\omega_2\omega_1}^2 \langle e_i, F_{\omega_1} v_j \rangle.$$

For the first term of the right-hand side, it suffices to estimate the tail of

$$(S_0)_{ij} \equiv p^{-1} \sum_{\omega_1} \xi_{\omega_1}^{(1)} E_{\omega_1} H_{\omega_1} \langle e_i, F_{\omega_1} v_j \rangle, \qquad H_{\omega_1} \equiv p^{-2} \sum_{\omega_2: \omega_2 \neq \omega_1} \xi_{\omega_2}^{(2)} P_{\omega_2\omega_1}^2,$$

where $\{\xi_\omega^{(1)}\}$ and $\{\xi_\omega^{(2)}\}$ are independent. We know from Section 6.3 that $\|H\|_\infty$ obeys $\|H\|_\infty \leq C \sqrt{\beta \log n} \, (\mu_0 nr/m)^{3/2}$ with probability at least $1 - 2n^{-\beta}$ for each $\beta > 2$. Thus, Lemma (6.12) shows that S_0 obeys (6.34)–(6.35) just as before. The other term is equal to $(1 - p)$ times $\sum_{\omega_1} E_{\omega_1} H_{\omega_1} \langle e_i, F_{\omega_1} v_j \rangle$, and by the Cauchy-Schwarz inequality and (6.4)

$$\left| \sum_{\omega_1} E_{\omega_1} H_{\omega_1} \langle e_i, F_{\omega_1} v_j \rangle \right| \leq \|H\|_\infty \|e_i^* E\| \left(\sum_b v_{bj}^2 \right)^{1/2} \leq C \sqrt{\frac{\mu_0}{n}} \sqrt{\beta \log n} \left(\frac{\mu_0 nr^{4/3}}{m} \right)^{3/2}$$

on the event where $\|H\|_\infty \leq C \sqrt{\beta \log n} \, (\mu_0 nr/m)^{3/2}$. Hence, when $m \geq \lambda \mu_0 \, nr^{4/3} \, (\beta \log n)$ for some numerical constant $\lambda > 0$, we have that $|\sum_{\omega_1} E_{\omega_1} H_{\omega_1} \langle e_i, F_{\omega_1} v_j \rangle| \leq \sqrt{\mu_0/n}$ on this event. In conclusion, the third term in (6.29) obeys (6.30) with probability at least $1 - O(n^{-\beta})$ provided that m is sufficiently large as above.

We proceed to the fourth term which can be written as

$$p^{-3}(1-2p) \sum_{\omega_1 \neq \omega_3} \xi_{\omega_1} \xi_{\omega_3} E_{\omega_3} P_{\omega_3\omega_1} P_{\omega_1\omega_1} \langle e_i, F_{\omega_1} v_j \rangle + p^{-2}(1-p) \sum_{\omega_1 \neq \omega_3} \xi_{\omega_3} E_{\omega_3} P_{\omega_3\omega_1} P_{\omega_1\omega_1} \langle e_i, F_{\omega_1} v_j \rangle.$$

We use the decoupling trick for the first term and bound the tail of

$$(S_0)_{ij} \equiv p^{-1} \sum_{\omega_1} \xi_{\omega_1}^{(1)} H_{\omega_1}(p^{-1} P_{\omega_1\omega_1}) \langle e_i, F_{\omega_1} v_j \rangle, \qquad H_{\omega_1} \equiv p^{-1} \sum_{\omega_3: \omega_3 \neq \omega_1} \xi_{\omega_3}^{(3)} E_{\omega_3} P_{\omega_3\omega_1},$$

where $\{\xi_\omega^{(1)}\}$ and $\{\xi_\omega^{(3)}\}$ are independent. We know from Section 6.2 that

$$\|H\|_\infty \leq C \sqrt{\frac{\mu_0 nr\beta \log n}{m}} \|E\|_\infty$$

with probability at least $1 - 2n^{-\beta}$ for each $\beta > 2$. Therefore, Lemma 6.12 shows that S_0 obeys (6.34)–(6.35) just as before. The other term is equal to $(1-p)$ times $\sum_{\omega_1} H_{\omega_1}(p^{-1} P_{\omega_1\omega_1}) \langle e_i, F_{\omega_1} v_j \rangle$, and the Cauchy-Schwarz inequality gives

$$\left| \sum_{\omega_1} H_{\omega_1}(p^{-1} P_{\omega_1\omega_1}) \langle e_i, F_{\omega_1} v_j \rangle \right| \leq \sqrt{n} \|H\|_\infty \frac{\mu_0 nr}{m} \leq C \frac{\mu_1 \sqrt{r\beta \log n}}{\sqrt{n}} \left(\frac{\mu_0 nr}{m} \right)^{3/2}$$

on the event $\|H\|_\infty \leq C \sqrt{\mu_0 nr(\beta \log n)/m} \|E\|_\infty$. Because $\mu_1 \leq \mu_0 \sqrt{r}$, we have that whenever $m \geq \lambda \mu_0^{4/3} nr^{5/3} (\beta \log n)$ for some numerical constant $\lambda > 0$, $p^{-1} |\sum_{\omega_1} H_{\omega_1} P_{\omega_1\omega_1} \langle e_i, F_{\omega_1} v_j \rangle| \leq$

$\sqrt{\mu_0/n}$ just as before. In conclusion, the fourth term in (6.29) obeys (6.30) with probability at least $1 - O(n^{-\beta})$ provided that m is sufficiently large as above.

We finally examine the last term

$$p^{-3} \sum_{\omega_1 \neq \omega_2 \neq \omega_3} \xi_{\omega_1} \xi_{\omega_2} \xi_{\omega_3} \, E_{\omega_3} P_{\omega_3 \omega_2} P_{\omega_2 \omega_1} \langle \boldsymbol{e}_i, \boldsymbol{F}_{\omega_1} \boldsymbol{v}_j \rangle.$$

Just as before, we need to bound the tail of

$$(\boldsymbol{S}_0)_{ij} \equiv p^{-1} \sum_{\omega_1, \omega_2, \omega_3} \xi_{\omega_1}^{(1)} H_{\omega_1} \langle \boldsymbol{e}_i, \boldsymbol{F}_{\omega_1} \boldsymbol{v}_j \rangle,$$

where $\boldsymbol{H}$ is given by (6.23). We know from Section 6.3 that $\boldsymbol{H}$ obeys

$$\|\boldsymbol{H}\|_\infty \leq C \, (\beta \log n) \, \frac{\mu_0 n r}{m} \, \mu_1 \frac{\sqrt{r}}{n}$$

with probability at least $1 - 4n^{-\beta}$ for each $\beta > 2$. Therefore, Lemma 6.12 gives

$$\mathbb{P}\left(\|\boldsymbol{S}_0\|_\infty \geq \frac{1}{5} \sqrt{\mu_0/n} \right) \leq 2n^2 e^{-1/D}, \quad D \leq C \left(\mu_0 \mu_1^2 (\beta \log n)^2 \left(\frac{nr}{m} \right)^3 + \mu_0 \mu_1 (\beta \log n) \left(\frac{nr}{m} \right)^2 \right)$$

for some $C > 0$. Hence, when $m \geq \lambda \mu_0 \, nr^{4/3} (\beta \log n)$ for some numerical constant $\lambda > 0$, we have that $\|\boldsymbol{S}_0\|_\infty \geq \frac{1}{5} \sqrt{\mu_0/n}$ with probability at most $2n^2 e^{-(\beta \log n)}$. In conclusion, the fifth term in (6.29) obeys (6.30) with probability at least $1 - O(n^{-\beta})$ provided that m is sufficiently large as above.

To summarize the calculations of this section, if $m = \lambda \, \mu_0^{4/3} n r^{5/3} \, (\beta \log n)$ where $\beta \geq 2$ is fixed and λ is some sufficiently large numerical constant, then

$$\sum_{1 \leq j \leq r} |\alpha_{ij}|^2 \leq \mu_0 r/n$$

with probability at least $1 - O(n^{-\beta})$. This concludes the proof. ∎

6.5   Proof of Lemma 4.8

It remains to study the spectral norm of $p^{-1}(\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E})$ for some positive integer k_0, which we bound by the Frobenius norm

$$p^{-1} \| (\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E}) \| \leq p^{-1} \| (\mathcal{P}_\Omega \mathcal{P}_T) \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E}) \|_F$$

$$\leq \sqrt{3/2p} \, \| \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E}) \|_F,$$

where the inequality follows from Corollary 4.3. To bound the Frobenius of the series, write

$$\| \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E}) \|_F \leq \|\mathcal{H}\|^{k_0} \|\boldsymbol{E}\|_F + \|\mathcal{H}\|^{k_0+1} \|\boldsymbol{E}\|_F + \dots$$

$$\leq \frac{\|\mathcal{H}\|^{k_0}}{1 - \|\mathcal{H}\|} \|\boldsymbol{E}\|_F.$$

Theorem 4.1 gives an upper bound on $\|\mathcal{H}\|$ since $\|\mathcal{H}\| \leq C_R \sqrt{\mu_0 n r \beta \log n / m} < 1/2$ on an event with probability at least $1 - 3n^{-\beta}$. Since $\|\boldsymbol{E}\|_F = \sqrt{r}$, we conclude that

$$p^{-1} \|(\mathcal{P}_\Omega \mathcal{P}_T) \sum_{k \geq k_0} \mathcal{H}^k(\boldsymbol{E})\|_F \leq C \frac{1}{\sqrt{p}} \left(\frac{\mu_0 n r \beta \log n}{m} \right)^{k_0/2} \sqrt{r} = C \left(\frac{n^2 r}{m} \right)^{1/2} \left(\frac{\mu_0 n r \beta \log n}{m} \right)^{k_0/2}$$

with large probability. This is the content of Lemma 4.8.

7   Numerical Experiments

To demonstrate the practical applicability of the nuclear norm heuristic for recovering low-rank matrices from their entries, we conducted a series of numerical experiments for a variety of the matrix sizes n, ranks r, and numbers of entries m. For each (n, m, r) triple, we repeated the following procedure 50 times. We generated $\boldsymbol{M}$, an $n \times n$ matrix of rank r, by sampling two $n \times r$ factors $\boldsymbol{M}_L$ and $\boldsymbol{M}_R$ with i.i.d. Gaussian entries and setting $\boldsymbol{M} = \boldsymbol{M}_L \boldsymbol{M}_R^*$. We sampled a subset Ω of m entries uniformly at random. Then the nuclear norm minimization

$$\begin{aligned} \text{minimize} \quad & \|\boldsymbol{X}\|_* \\ \text{subject to} \quad & X_{ij} = M_{ij}, \quad (i,j) \in \Omega \end{aligned}$$

was solved using the SDP solver SDPT3 [34]. We declared $\boldsymbol{M}$ to be recovered if the solution returned by the SDP, $\boldsymbol{X}_{\mathbf{opt}}$, satisfied $\|\boldsymbol{X}_{\mathbf{opt}} - \boldsymbol{M}\|_F / \|\boldsymbol{M}\|_F < 10^{-3}$. Figure 1 shows the results of these experiments for $n = 40$ and 50. The x-axis corresponds to the fraction of the entries of the matrix that are revealed to the SDP solver. The y-axis corresponds to the ratio between the dimension of the set of rank r matrices, $d_r = r(2n - r)$, and the number of measurements m. Note that both of these axes range from zero to one as a value greater than one on the x-axis corresponds to an overdetermined linear system where the semidefinite program always succeeds, and a value of greater than one on the y-axis corresponds to a situation where there is always an infinite number of matrices with rank r with the given entries. The color of each cell in the figures reflects the empirical recovery rate of the 50 runs (scaled between 0 and 1). White denotes perfect recovery in all experiments, and black denotes failure for all experiments. Interestingly, the experiments reveal very similar plots for different n, suggesting that our asymptotic conditions for recovery may be rather conservative.

For a second experiment, we generated random *positive semidefinite* matrices and tried to recover them from their entries using the nuclear norm heuristic. As above, we repeated the same procedure 50 times for each (n, m, r) triple. We generated $\boldsymbol{M}$, an $n \times n$ positive semidefinite matrix of rank r, by sampling an $n \times r$ factor $\boldsymbol{M}_F$ with i.i.d. Gaussian entries and setting $\boldsymbol{M} = \boldsymbol{M}_F \boldsymbol{M}_F^*$. We sampled a subset Ω of m entries uniformly at random. Then we solved the nuclear norm minimization problem

$$\begin{aligned} \text{minimize} \quad & \text{trace}(\boldsymbol{X}) \\ \text{subject to} \quad & X_{ij} = M_{ij}, \quad (i,j) \in \Omega \;. \\ & \boldsymbol{X} \succeq 0 \end{aligned}$$

As above, we declared $\boldsymbol{M}$ to be recovered if $\|\boldsymbol{X}_{\mathbf{opt}} - \boldsymbol{M}\|_F / \|\boldsymbol{M}\|_F < 10^{-3}$. Figure 2 shows the results of these experiments for $n = 40$ and 50. The x-axis again corresponds to the fraction of the entries of the matrix that are revealed to the SDP solver, but, in this case, the number of
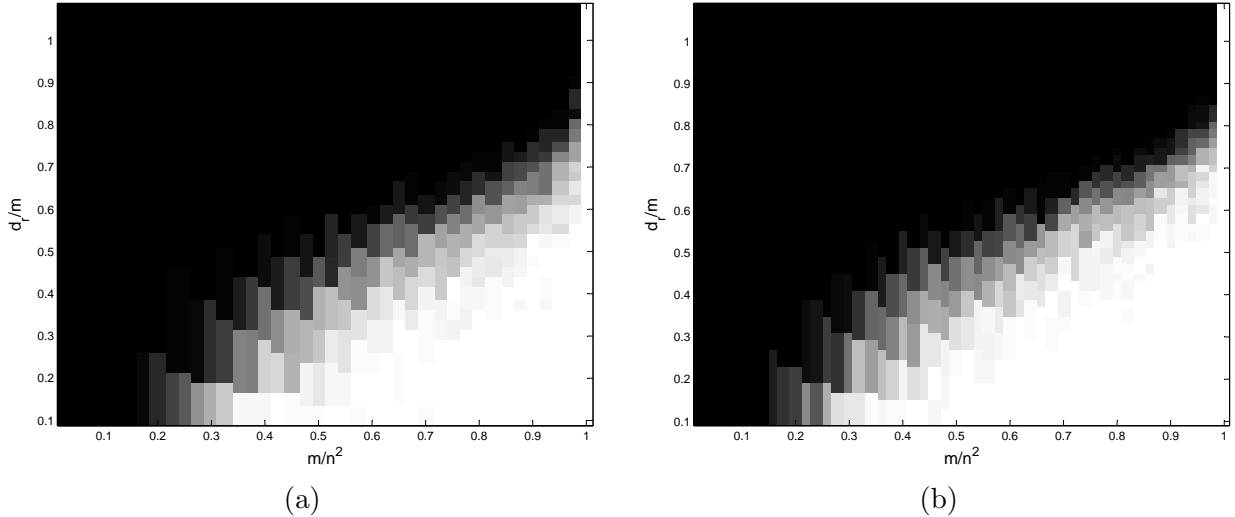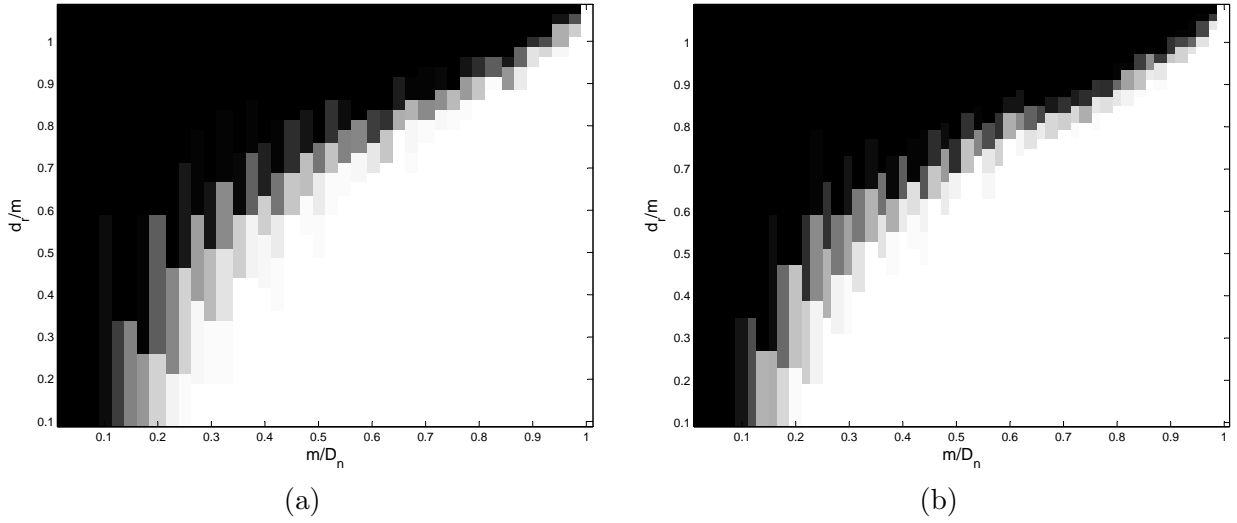
Figure 1: Recovery of full matrices from their entries. For each (n, m, r) triple, we repeated the following procedure 50 times. A matrix $\boldsymbol{M}$ of rank r and a subset of m entries were selected at random. Then we solved the nuclear norm minimization for $\boldsymbol{X}$ subject to $X_{ij} = M_{ij}$ on the selected entries. We declared $\boldsymbol{M}$ to be recovered if $\|\boldsymbol{X}_{\mathbf{opt}} - \boldsymbol{M}\|_F / \|\boldsymbol{M}\|_F < 10^{-3}$. The results are shown for (a) $n = 40$ and (b) $n = 50$. The color of each cell reflects the empirical recovery rate (scaled between 0 and 1). White denotes perfect recovery in all experiments, and black denotes failure for all experiments.

measurements is divided by $D_n = n(n+1)/2$, the number of unique entries in a positive-semidefinite matrix and the dimension of the rank r matrices is $d_r = nr - r(r-1)/2$. The color of each cell is chosen in the same fashion as in the experiment with full matrices. Interestingly, the recovery region is much larger for positive semidefinite matrices, and future work is needed to investigate if the theoretical scaling is also more favorable in this scenario of low-rank matrix completion.

Finally, in Figure 3, we plot the performance of the nuclear norm heuristic when recovering low-rank matrices from Gaussian projections of these matrices. In these cases, $\boldsymbol{M}$ was generated in the same fashion as above, but, in place of sampling entries, we generated m random Gaussian projections of the data (see the discussion in Section 1.4). Then we solved the optimization

$$
\begin{array}{ll}
\text{minimize} & \|\boldsymbol{X}\|_* \\
\text{subject to} & \mathcal{A}(\boldsymbol{X}) = \mathcal{A}(\boldsymbol{M})
\end{array}
$$

with the additional constraint that $\boldsymbol{X} \succeq 0$ in the positive semidefinite case. Here $\mathcal{A}(\boldsymbol{X})$ denotes a linear map of the form (1.15) where the entries are sampled i.i.d. from a zero-mean unit variance Gaussian distribution. In these experiments, the recovery regime is far larger than in the case of that of sampling entries, but this is not particularly surprising as each Gaussian observation measures a contribution from every entry in the matrix $\boldsymbol{M}$. These Gaussian models were studied extensively in [27].

Figure 2: Recovery of positive semidefinite matrices from their entries. For each (n, m, r) triple, we repeated the following procedure 50 times. A positive semidefinite matrix $\boldsymbol{M}$ of rank r and a set of m entries were selected at random. Then we solved the nuclear norm minimization subject to $X_{ij} = M_{ij}$ on the selected entries with the constraint that $\boldsymbol{X} \succeq 0$. The color scheme for each cell denotes empirical recovery probability and is the same as in Figure 1. The results are shown for (a) $n = 40$ and (b) $n = 50$.
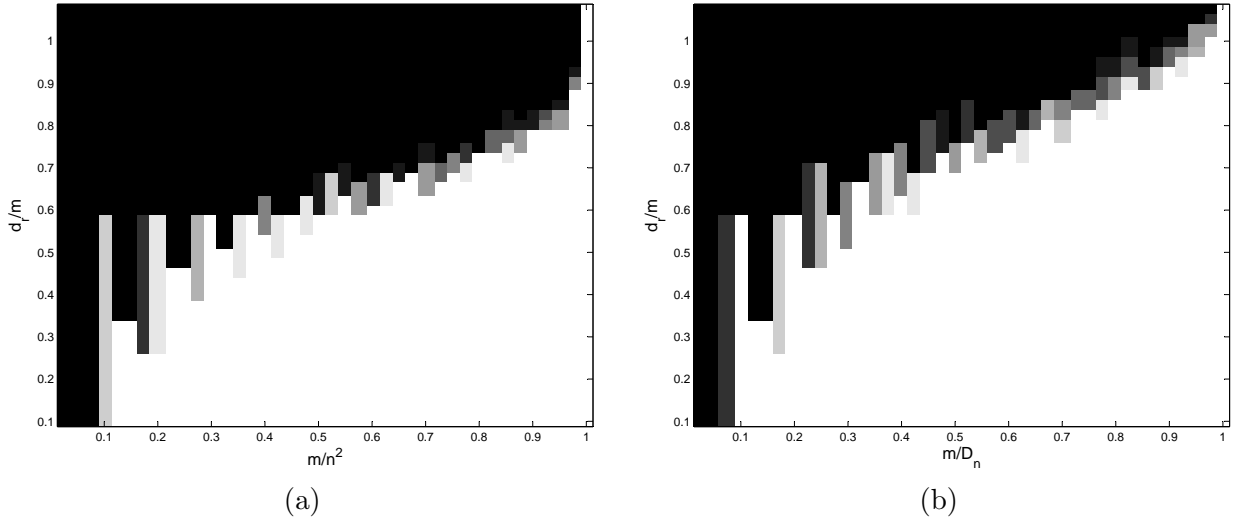
Figure 3: Recovery of matrices from Gaussian observations. For each (n, m, r) triple, we repeated the following procedure 10 times. In (a), a matrix of rank r was generated as in Figures 1. In (b) a positive semidefinite matrix of rank r was generated as in Figures 2. In both plots, we select a matrix $\mathcal{A}$ from the Gaussian ensemble with m rows and n^2 (in (a)) or $D_n = n(n + 1)/2$ (in (b)) columns. Then we solve the nuclear norm minimization subject to $\mathcal{A}(\boldsymbol{X}) = \mathcal{A}(\boldsymbol{M})$. The color scheme for each cell denotes empirical recovery probability and is the same as in Figures 1 and 2.

8 Discussion

8.1 Improvements

In this paper, we have shown that under suitable conditions, one can reconstruct an $n \times n$ matrix of rank r from a small number of its sampled entries provided that this number is on the order of $n^{1.2} r \log n$, at least for moderate values of the rank. One would like to know whether better results hold in the sense that exact matrix recovery would be guaranteed with a reduced number of measurements. In particular, recall that an $n \times n$ matrix of rank r depends on $(2n - r)r$ degrees of freedom; is it true then that it is possible to recover most low-rank matrices from on the order of nr—up to logarithmic multiplicative factors—randomly selected entries? Can the sample size be merely proportional to the true complexity of the low-rank object we wish to recover?

In this direction, we would like to emphasize that there is nothing in our approach that apparently prevents us from getting stronger results. Indeed, we developed a bound on the spectral norm of each of the first four terms $(\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \mathcal{H}^k(E)$ in the series (4.13) (corresponding to values of k equal to $0, 1, 2, 3$) and used a general argument to bound the remainder of the series. Presumably, one could bound higher order terms by the same techniques. Getting an appropriate bound on $\|(\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \mathcal{H}^4(E)\|$ would lower the exponent of n from $6/5$ to $7/6$. The appropriate bound on $\|(\mathcal{P}_{T^\perp} \mathcal{P}_\Omega \mathcal{P}_T) \mathcal{H}^5(E)\|$ would further lower the exponent to $8/7$, and so on. To obtain an optimal result, one would need to reach k of size about $\log n$. In doing so, however, one would have to pay special attention to the size of the decoupling constants (the constant C_D for two variables in Lemma 6.5) which depend on k—the number of decoupled variables. These constants grow with k and upper bounds are known [15, 16].

8.2 Further directions

It would be of interest to extend our results to the case where the unknown matrix is approximately low-rank. Suppose we write the SVD of a matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = \sum_{1 \leq k \leq n} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*,$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ and assume for simplicity that none of the σ_k's vanish. In general, it is impossible to complete such a matrix exactly from a partial subset of its entries. However, one might hope to be able to recover a good approximation if, for example, most of the singular values are small or negligible. For instance, consider the truncated SVD of the matrix $\boldsymbol{M}$,

$$\boldsymbol{M}_r = \sum_{1 \leq k \leq r} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^*,$$

where the sum extends over the r largest singular values and let $\boldsymbol{M}_\star$ be the solution to (1.5). Then one would not expect to have $\boldsymbol{M}_\star = \boldsymbol{M}$ but it would be of great interest to determine whether the size of $\boldsymbol{M}_\star - \boldsymbol{M}$ is comparable to that of $\boldsymbol{M} - \boldsymbol{M}_r$ provided that the number of sampled entries is sufficiently large. For example, one would like to know whether it is reasonable to expect that $\|\boldsymbol{M}_\star - \boldsymbol{M}\|_*$ is on the same order as $\|\boldsymbol{M} - \boldsymbol{M}_r\|_*$ (one could ask for a similar comparison with a different norm). If the answer is positive, then this would say that approximately low-rank matrices can be accurately recovered from a small set of sampled entries.

Another important direction is to determine whether the reconstruction is robust to noise as in some applications, one would presumably observe

$$Y_{ij} = M_{ij} + z_{ij}, \quad (i,j) \in \Omega,$$

where z is a deterministic or stochastic perturbation. In this setup, one would perhaps want to minimize the nuclear norm subject to $\|\mathcal{P}_\Omega(\boldsymbol{X} - \boldsymbol{Y})\|_F \le \epsilon$ where ϵ is an upper bound on the noise level instead of enforcing the equality constraint $\mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y})$. Can one expect that this algorithm or a variation thereof provides accurate answers? That is, can one expect that the error between the recovered and the true data matrix be proportional to the noise level?

9 Appendix

9.1 Proof of Theorem 4.2

The proof of (4.10) follows that in [10] but we shall use slightly more precise estimates.

Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables taking values in a Banach space and let $Y_\star$ be the supremum defined as

$$Y_\star = \sup_{f \in \mathcal{F}} \sum_{i=1}^n f(Y_i), \tag{9.1}$$

where $\mathcal{F}$ is a countable family of real-valued functions such that if $f \in \mathcal{F}$, then $-f \in \mathcal{F}$. Talagrand [33] proved a concentration inequality about $Y_\star$, see also [22, Corollary 7.8].

Theorem 9.1 *Assume that $|f| \le B$ and $\mathbb{E} f(Y_i) = 0$ for every f in $\mathcal{F}$ and $i = 1, \ldots, n$. Then for all $t \ge 0$,*

$$\mathbb{P}(|Y_\star - \mathbb{E} Y_\star| > t) \le 3 \exp\left(-\frac{t}{KB} \log\left(1 + \frac{Bt}{\sigma^2 + B \mathbb{E} Y_\star}\right)\right), \tag{9.2}$$

where $\sigma^2 = \sup_{f \in \mathcal{F}} \sum_{i=1}^n \mathbb{E} f^2(Y_i)$, and K is a numerical constant.

We note that very precise values of the numerical constant K are known and are small, see [20].

We will apply this theorem to the random variable Z defined in the statement of Theorem 4.2. Put $\mathcal{Y}_{ab} = p^{-1}(\delta_{ab} - p)\,\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*) \otimes \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)$ and $\mathcal{Y} = \sum_{ab} \mathcal{Y}_{ab}$. By definition,

$$Z = \sup \, \langle \boldsymbol{X}_1, \mathcal{Y}(\boldsymbol{X}_2)\rangle = \sup \, \sum_{ab}\langle \boldsymbol{X}_1, \mathcal{Y}_{ab}(\boldsymbol{X}_2)\rangle$$

$$= \sup \, p^{-1}\sum_{ab}(\delta_{ab} - p)\langle \boldsymbol{X}_1, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle \, \langle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*), \boldsymbol{X}_2\rangle,$$

where the supremum is over a countable collection of matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ obeying $\|\boldsymbol{X}_1\|_F \le 1$ and $\|\boldsymbol{X}_2\|_F \le 1$. Note that it follows from (4.8)

$$|\langle \boldsymbol{X}_1, \mathcal{Y}_{ab}(\boldsymbol{X}_2)\rangle| = p^{-1}\,|\delta_{ab} - p|\,|\langle \boldsymbol{X}_1, \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\rangle|\,|\langle \mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*), \boldsymbol{X}_2\rangle|$$

$$\le p^{-1}\,\|\mathcal{P}_T(\boldsymbol{e}_a\boldsymbol{e}_b^*)\|_F^2 \le 2\mu_0 r/(\min(n_1, n_2)p) = 2\mu_0\,nr/m$$

(recall that $n = \max(n_1, n_2)$). Hence, we can apply Theorem 9.1 with $B = 2\mu_0(nr/m)$. Also

$$\mathbb{E} |\langle \boldsymbol{X}_1, \mathcal{Y}_{ab}(\boldsymbol{X}_2) \rangle|^2 = p^{-1}(1-p) |\langle \boldsymbol{X}_1, \mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*) \rangle|^2 |\langle \boldsymbol{X}_2, \mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*) \rangle|^2$$
$$\leq p^{-1} \|\mathcal{P}_T(\boldsymbol{e}_a \boldsymbol{e}_b^*)\|_F^2 |\langle \mathcal{P}_T(\boldsymbol{X}_2), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle|^2$$

so that

$$\sum_{ab} \mathbb{E} |\langle \boldsymbol{X}_1, \mathcal{Y}_{ab}(\boldsymbol{X}_2) \rangle|^2 \leq (2\mu_0 \, nr/m) \sum_{ab} |\langle \mathcal{P}_T(\boldsymbol{X}_2), \boldsymbol{e}_a \boldsymbol{e}_b^* \rangle|^2$$
$$= (2\mu_0 \, nr/m) \|\mathcal{P}_T(\boldsymbol{X}_2)\|_F^2 \leq 2\mu_0 nr/m.$$

Since $\mathbb{E} Z \leq 1$, Theorem 9.1 gives

$$P(|Z - \mathbb{E} Z| > t) \leq 3 \exp\left(-\frac{t}{KB} \log(1 + t/2)\right) \leq 3 \exp\left(-\frac{t \log 2}{KB} \min(1, t/2)\right),$$

where we have used the fact that $\log(1+u) \geq (\log 2) \min(1, u)$ for $u \geq 0$. Plugging $t = \lambda \sqrt{\frac{\mu_0 \, nr \log n}{m}}$ and $B = 2\mu_0 \, nr/m$ establishes the claim.

9.2   Proof of Lemma 6.2

We shall make use of the following lemma which is an application of well-known deviation bounds about binomial variables.

Lemma 9.2 *Let $\{\delta_i\}_{1 \leq i \leq n}$ be a sequence of i.i.d. Bernoulli variables with $\mathbb{P}(\delta_i = 1) = p$ and $Y = \sum_{i=1}^n \delta_i$. Then for each $\lambda > 0$,*

$$\mathbb{P}(Y > \lambda \, \mathbb{E} Y) \leq \exp\left(-\frac{\lambda^2}{2 + 2\lambda/3} \, \mathbb{E} Y\right). \tag{9.3}$$

The random variable $\sum_b \delta_{ab} E_{ab}^2$ is bounded by $\|\boldsymbol{E}\|_\infty^2 \sum_b \delta_{ab}$ and it thus suffices to estimate the qth moment of $Y_* = \max Y_a$ where $Y_a = \sum_b \delta_{ab}$. The inequality (9.3) implies that

$$\mathbb{P}(Y_* > \lambda np) \leq n \exp\left(-\frac{\lambda^2}{2 + 2\lambda/3} \, np\right),$$

and for $\lambda \geq 2$, this gives $\mathbb{P}(Y_* > \lambda np) \leq n \, e^{-\lambda np/2}$. Hence

$$\mathbb{E} Y_*^q = \int_0^\infty \mathbb{P}(Y_* > t) \, qt^{q-1} \, dt \leq (2np)^q + \int_{2np}^\infty n \, e^{-t/2} \, qt^{q-1} \, dt.$$

By integrating by parts, one can check that when $q \leq np$, we have

$$\int_{2np}^\infty n \, e^{-t/2} \, qt^{q-1} \, dt \leq nq \, (2np)^q \, e^{-np}.$$

Under the assumptions of the lemma, we have $nq \, e^{-np} \leq 1$ and, therefore,

$$\mathbb{E} Y_*^q \leq 2 \, (2np)^q.$$

The conclusion follows.

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