CS 830: Randomness and Complexity 5/5/05

Lecture 30: Undirected st-Connectivity in Log-Space (Continued)

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1 Overview

Recall from last time, we had an undirected graph G with N nodes of regular degree D. A was the normalized adjacency matrix for this graph. If G is D-regular then $|\lambda_i| \leq D$ and D is the maximum eigenvalue whose corresponding eigenvector is the all ones vector (1...1). In the normalized adjacency matrix the eigenvalue of D corresponds to the eigenvalue 1. In the normalized matrix the eigenvalue -1 is also possible, but only if G is bipartite. We can force G to be non-bipartite G by adding self-loops. $\lambda = max_{\lambda_i \neq 1}|\lambda_i|$ is the second largest eigenvalue of this graph. It can be shown that $\lambda < 1 - \frac{1}{Dn^2}$.

We have another d-regular graph, H, with D nodes, whose normalized adjacency matrix is B and whose second largest eigenvalue is μ.

2 The Zig-Zag Product

We will use the notation $G \oslash H$ to represent the zig-zag product between G and H. $G \oslash H$ is defined as follows. For all $v \in V(G)$, replace v by a copy of H called H_v. Then $|V(G \oslash H)| = |V(G) \otimes V(H)| = N \cdot D$. If we have:

$$\begin{aligned} Rot_H(a,i) &= (a',i'), i \in [d] \\ Rot_G(u,a') &= (v,b') \\ Rot_H(b',j) &= (b,j'), j \in [d] \end{aligned} \tag{1}$$

then

$$Rot_{G \oslash H}[(u,a),(i,j)] = [(v,b),(j',i')] \tag{2}$$

with the rotation map, Rot, was define as it was last lecture. See Figure 1 on the next page for a diagram of this definition.

3 Reingold's Algorithm

Define G_i recurrently as follows:

$$\begin{aligned} G_0 &= G \\ G_1 &= (G_0 \oslash H)^c \\ G_2 &= (G_1 \oslash H)^c \\ &... \\ G_i &= (G_{i-1} \oslash H)^c \end{aligned} \tag{3}$$

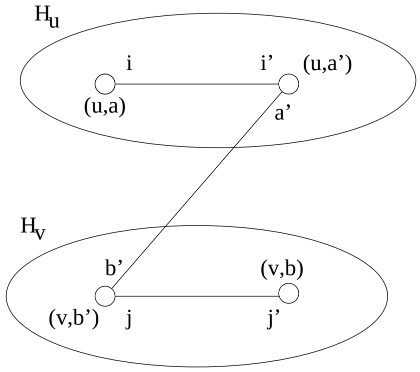

Figure 1: Diagram of rotation mapping on $G \oslash H$

We will arrange that $D = d^{2c}$, so that $deg(G_1) = (d^2)^c = D$. For any adjacency matrix X the following holds (where $\lambda(Y)$ is the second largest eigenvalue of adjacency matrix Y): $\lambda(X^k) = [\lambda(X)]^k$. So if $\lambda < (1+\frac{1}{Dn^2})$ we need $k \sim n^2$ to get λ exponentially small. We need to iterate about $k \sim log(n^2) = O(log(n))$ times, squaring at each step.

It also helps to consider breaking the algorithm down into further steps: $G_0 \to G_{1/2} \to G_1 \to G_{3/2} \to ....$ Where for the step from G_i to $G_{i+1/2}$, for integral i, lowers the degree of the graph, while trying to maintain the eigenvalue gap. The step from G_i to $G_{i+1/2}$, for non-integral i, should increase the degree of the graph and increase the eigenvalue gap.

The bound on the eigenvalue gap that Reingold shows using the zig-zag product construction is: $1 - \lambda(G \oslash H) \geq \frac{1}{2}(1 - \lambda(G))(1 - \lambda(H)^2)$. The second term can be large ($(1 - \lambda(H)^2) \sim 1$). So we get $\frac{1}{2}(1 - \lambda(G \oslash H)) \approx \frac{1}{2}(1 - \lambda(G))$. We can run the algorithm for $l = O(log(n))$ steps to get $\lambda(G_l) \leq \frac{1}{2}$. We can construct this graph in only $log(n)$ space. Then we will have enough to perform exhaustive search for st-connectivity as we mentioned last lecture.

4 Proving the Eigenvalue Bound of Zig-Zag Products

Define a 0-1 matrix $\tilde{A}$: $\tilde{A}[(u,a),(v,b)] = 1$ if and only if $Rot_G(u,a) = (v,b)$ and zero otherwise. $\tilde{A}$ represents the "zag" step of the algorithm. $\tilde{A}$ is a $ND \times ND$ permutation matrix and is symmetric and orthogonal, $\tilde{A}^2 = I$.

Define a matrix $\tilde{B}$: $\tilde{B} = I_n \otimes B$. Where I_n is the n dimensional identity matrix and B is still the adjacency matrix for H.

Then, the normalized adjacency matrix for $G \oslash H$ is $\tilde{B}\tilde{A}\tilde{B}$. We would like to estimate the second eigenvalue of the adjacency matrix in its new form.

Consider any $w \in \mathbb{R}^{N \cdot D}$ such that $w \perp \vec{1}_{ND}$. An alternate definition for λ:

$$max_{w \perp \vec{1}_{ND}} \left| \frac{\langle \tilde{B}\tilde{A}\tilde{B}, w \rangle}{\langle w, w \rangle} \right| = \lambda(G \oslash H) \tag{4}$$

We can rewrite $\langle \tilde{B}\tilde{A}\tilde{B}, w \rangle$ as $\langle \tilde{A}\tilde{B}, \tilde{B}w \rangle$ because B is symmetric and thus so is $\tilde{B}$.

Define $\alpha \in \mathbb{R}^{N \cdot D}, \alpha \perp \vec{1}_{ND}$, where $\vec{1}_x$ is the x-dimensional vector containing all ones. Think of α this as a weight distribution. $\alpha = (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{R}^D$.

Define $\alpha^s = (\sum_{s=1}^{D} \alpha_{1s}, \sum_{s=1}^{D} \alpha_{2s}, ..., \sum_{s=1}^{D} \alpha_{Ns}) \in \mathbb{R}^N$. Since $\alpha \perp \vec{1}_{ND}$, $\alpha^s \perp \vec{1}_N$.

Given a vector $\gamma \in \mathbb{R}^N$ (think of this as a weight distribution on the original G), we can define a "spreading out" of γ:

$$\gamma^e = (\frac{\gamma_1}{D}, ..., \frac{\gamma_1}{D}, \frac{\gamma_2}{D}, ..., \frac{\gamma_2}{D}, ..., \frac{\gamma_N}{D}, ..., \frac{\gamma_N}{D}) = \frac{\gamma \otimes \vec{1}_D}{D}. \tag{5}$$

Observe that $(\gamma^e)^s) = \gamma$ and that $\alpha^{ses} = \alpha^s$.

LEMMA 4.1 *For all γ on G, $(\tilde{A}\gamma^e)^s = A\gamma$.*

Proof:

$$\begin{aligned}
((\tilde{A}\gamma^e)^s)_v &= \sum_{b=1}^{D} (\tilde{(A)}\gamma^e)_{(v,b)} \\
&= \sum_{b=1}^{D} \sum_{(u,a)} (\tilde{A}_{(u,a),(v,b)}\gamma^e)_{(u,a)} \\
&= \sum_{b=1}^{D} \gamma^e_{(u,a)}, Rot_G(v,b) = (u,a) \qquad (6) \\
&= \frac{1}{D} \sum_{b=1}^{D} \gamma_u \\
&= \sum_{v \text{ neighbors } u} A_{vu}\gamma_u \\
&= (A\gamma)_v.
\end{aligned}$$

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LEMMA 4.2 *If $x = (x_1, ..., x_N) \in \mathbb{R}^{N \cdot D}$, where $x_i \in \mathbb{R}^D$ then $\langle x, y \otimes \vec{1}_D \rangle = \langle x^s, y \rangle$, where $y \in \mathbb{R}^N$.*

Proof:

$$\begin{aligned}
x &= (x_{11}, ..., x_{1D}, x_{21}, ..., x_{2D}, ..., x_{N1}, ..., x_{NN}), x_{ij} \in \mathbb{R} \\
y \otimes \vec{1}_D &= (y_1, ..., y_1, y_2, ..., y_2, ..., y_N, ..., y_N) \\
\langle x, y \otimes \vec{1}_D \rangle &= \sum_{i=1}^{N} y_i \sum_{j=1}^{D} x_{ij} \\
&= \sum_{j=1}^{D} \sum_{i=1}^{N} x_{ij} y_i \qquad (7) \\
&= \sum_{i=1}^{N} x_i^s y_i \\
&= \langle x^s, y \rangle
\end{aligned}$$

$\blacksquare$

Define $V = \frac{1}{D} \cdot J$ where J is the $D \times D$ matrix of all ones. Then we have V commutes with B ($BV = VB$). Define $\alpha_i^{||} = \alpha_i^{se} = (\frac{\sum_{s=1}^{D} \alpha_{is}}{D}, ..., \frac{\sum_{s=1}^{D} \alpha_{is}}{D})$. This is the projection of α_i onto $\vec{1}_D$. Define $\alpha_i^{\perp} = \alpha_i - \alpha_i^{||}$. Then consequently, $\alpha^{||} = (\alpha_1^{||}, ..., \alpha_N^{||})$ and $\alpha^{\perp} = (\alpha_1^{\perp}, ..., \alpha_N^{\perp})$. So $\alpha = \alpha^{||} + \alpha^{\perp}$ and $\|\alpha\|^2 = \|a^{||}\|^2 + \|\alpha^{\perp}\|^2$.

Now, back to the original problem of estimating $\langle \tilde{A}\tilde{B}\alpha, \tilde{B}\alpha \rangle$. We have $\tilde{B}\alpha = \tilde{B}\alpha^{||} + \tilde{B}\alpha^{\perp}$ by linearity. $\tilde{B}\alpha^{||} = \alpha^{||}$ since $\vec{1}_D$ is the eigenvector corresponding to the eigenvalue 1. We can expand the inner product using the perpendicular and parallel components of α.

$$
\begin{aligned}
\langle \tilde{A}\tilde{B}\alpha, \tilde{B}\alpha \rangle = {} & \langle \tilde{A}\alpha^{||}, \alpha^{||} \rangle, (i) \\
& + \langle \tilde{A}\tilde{B}\alpha^{\perp}, \alpha^{||} \rangle, (ii) \\
& + \langle \tilde{A}\alpha^{||}, \tilde{B}\alpha^{\perp} \rangle, (iii) \\
& + \langle \tilde{A}\tilde{B}\alpha^{\perp}, \tilde{B}\alpha^{\perp} \rangle, (iv)
\end{aligned} \tag{8}
$$

The last term in the equation (iv) is less than $\mu^2 \|\alpha^{\perp}\|^2$ by the Cauchy-Schwartz inequality:

$$
\begin{aligned}
\langle \tilde{A}\tilde{B}\alpha^{\perp}, \tilde{B}\alpha^{\perp} \rangle &= \sum_{i=1}^{N} \|\tilde{B}\alpha_i^{\perp}\|^2 \\
&\leq \sum_{i=1}^{N} \mu^2 \|\alpha_i^{\perp}\|^2 \\
&\leq \mu^2 \|\alpha^{\perp}\|^2.
\end{aligned} \tag{9}
$$

The cross term $(ii) + (iii)$ are less than

$$
\begin{aligned}
&\leq 2\|\tilde{B}\alpha^{\perp}\| \|\alpha^{||}\| \\
&\leq 2\mu \|\alpha^{\perp}\| \|\alpha^{||}\| \\
&\leq \mu[\|\alpha^{||}\|^2 + \|\alpha^{\perp}\|^2] \\
&= \mu \|\alpha\|^2.
\end{aligned} \tag{10}
$$

Finally the term (i), using everything we have set up for the expansion and contraction of vectors:

$$
\begin{aligned}
\langle \tilde{A}\alpha^{||}, \alpha^{||} \rangle &= \langle \tilde{A}(\alpha^{se}), \alpha^{se} \rangle \\
&= \langle \tilde{A}(\alpha^{se}), \frac{\alpha \otimes \vec{1}_D}{D} \rangle \\
&= \frac{1}{D} \langle (\tilde{A}(\alpha^{se}))^s, \alpha^s \rangle \\
&= \frac{1}{D} \langle A\alpha^s, \alpha^s \rangle \\
|\frac{1}{D} \langle A\alpha^s, \alpha^s \rangle| &= \frac{1}{D} \lambda \|\alpha^s\|^2 \\
&= \lambda \|\alpha^{||}\|^2.
\end{aligned} \tag{11}
$$

Thus we have:

$$\langle \tilde{A}\tilde{B}\alpha, \tilde{B}\alpha\rangle \leq \lambda\|\alpha^{\|}\|^2 + \mu\|\alpha\|^2 + \mu^2\|\alpha^{\perp}\|^2 \leq (\lambda + \mu + \mu^2)\|\alpha\|^2. \tag{12}$$

Reingold actually proved several other things as well, using similar techniques.

Theorem 4.3 *Given G and H as specified then:*

1. $\lambda(G \oslash H) \leq \lambda + \mu + \mu^2$,
2. $\lambda(G \oslash H) \leq \sqrt{\lambda^2 + \mu^2 + \lambda\mu + \mu^4}$,
3. $1 - \lambda(G \oslash H) \geq \frac{\lambda\mu}{8}$.

This concludes CS 830.