THE abc-CONJECTURE AND THE LARGEST PRIME FACTOR OF $2^n + 1$

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ABSTRACT. We show that under the abc-conjecture the largest prime factor of $2^n + 1$ is $\Omega(n^{\frac{4}{3}})$ infinitely often.

1. Main Result

The abc-conjecture. (Oesterlé, Masser, Szpiro) For any $\epsilon > 0$, and $a, b, c \in \mathbb{Z}$ which are relatively prime satisfying $a + b = c$, we have

$$\max\{|a|, |b|, |c|\} \ll_\epsilon \left(\prod_{p \backslash abc} p \right)^{1+\epsilon}.$$

This conjecture has some amazing consequences for example it implies the Fermat's Last Theorem for all large enough exponents. In this article we show that it yields some non-trivial lower bounds on the largest prime factor of the sequence $2^n + 1$.

Theorem 1.1. *The largest prime factor of $2^n + 1$ is $\Omega(n^{\frac{4}{3}})$ infinitely often.*

Proof : Let P_x be the largest prime factor of

$$\prod_{1 \le n \le x} (2^n + 1).$$

We will adopt the following notation:

$$\square_n = \prod_{p \backslash\backslash 2^n+1} p$$

$$\blacksquare_n = \prod_{p^\nu \backslash\backslash 2^n+1, \nu \ge 2} p^\nu$$

The abc-conjecture can be used to lower bound the product

$$\prod_{1 \le n \le x} \square_n.$$

Since $2^n, 1$ and $2^n + 1$ are relatively prime in pairs and since $2^n + 1 = \square_n \blacksquare_n$ we have for any fixed $\epsilon > 0$,

$$\begin{aligned} 2^n + 1 &= \max\{2^n, 1, 2^n + 1\} \\ &\ll_\epsilon \left(2\square_n \sqrt{\blacksquare_n} \right)^{1+\epsilon} \end{aligned}$$

since $\blacksquare_n$ is the powerful part of $2^n + 1$. Since $\blacksquare_n \le 2^n + 1$, we get

$$\square_n \gg 2^{\frac{n(1-\epsilon)}{2}}.$$

Thus we get

$$\log \left(\prod_{1 \le n \le x} \square_n \right) \gg_\epsilon x^2.$$

We also have

$$\prod_{1\le n\le x} \square_n = \prod_{p\le P_x} p^{N_x(p)}$$

where

$$N_x(p) = \sum_{1\le n\le x,\, p\backslash\backslash 2^n+1} 1.$$

Putting these together we get the lower bound

$$\sum_{p\le P_x} N_x(p)\log p \gg_\epsilon x^2.$$

Now we will derive an upper bound for the sum $\sum_{p\le P_x} N_x(p)\log p$ in terms of P_x to get the result. We split up $\sum_{p\le P_x} N_x(p)\log p$ into two parts $\Sigma_A = \sum_{p\le x} N_x(p)\log p$ and $\Sigma_B = \sum_{x\le p\le P_x} N_x(p)\log p$.

Estimation of Σ_A: If $2^i + 1 \equiv 0 \mod p$, we have $i \ge \log p$ this is in particular a lower bound on the order of 2 in the multiplicative subgroup of p. Thus we have $N_x(p) \le \frac{x}{\log p} + 1$. Now

$$\begin{aligned}\Sigma_A &= \sum_{p\le x} N_x(p)\log p \\ &\ll \sum_{p\le x}\left(\frac{x\log p}{\log p} + \log p\right) \\ &= O\left(\frac{x^2}{\log x}\right).\end{aligned}$$

Estimation of Σ_B:
We can assume that

$$\forall p\ :\ x\le p\le P_x\ :\ \mathrm{Ord}_p(2) \ge \sqrt{p}$$

since otherwise the theorem is already true. For $p \ge x$ we have $N_x(p) \le \frac{\varphi(p)}{\sqrt{p}} \le \sqrt{p}$. Thus

$$\begin{aligned}\Sigma_B &= \sum_{x\le p\le P_x} N_x(p)\log p \\ &\le \sum_{x\le p\le P_x} \sqrt{p}\log p \\ &\le \sqrt{P_x}\sum_{x\le p\le P_x}\log p \\ &\le P_x^{1.5}.\end{aligned}$$

Together with the lower bound for $\Sigma_A + \Sigma_B$ we have

$$P_x^{1.5} \gg_\epsilon x^2.$$

This proves the theorem. □

Remark 1.2. The main use of the abc-conjecture was to avoid the estimation of the sum

$$\sum_{p\le P_x,\, p^a\backslash 2^n+1,\, a>1} N_x(p^a)\log p.$$

It seems like good upper bounds on the number of *Wieferich primes* (primes p such that $2^{p-1}\equiv 1 \mod p^2$) are required to bound this sum. Though only two Wieferich primes are known below 3×10^9 its still an open problem whether there are infinitely many such primes (or even if there are infinitely many non-Wieferich primes). Under the abc-conjecture we know that there are infinitely many non-Wieferich primes [Sil88] but the lower bound is only logarithmic.

References

[Sil88] Silverman, Joseph H.; *Wieferich's criterion and the* abc*-conjecture*, J. Number Theory, (**30**), no. 2, 226-237, (1988).