VANISHING OF COHOMOLOGY GROUPS OF AFFINE SCHEMES - (I)

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Throughout by rings we mean rings that have the identity element and homomorphisms will preserve the identity.

1. Differential Modules

We call an A-module X a *differential module* if X is equiped with an A-module endomorphism $d : X \to X$ such that $d^2 = 0$. Suppose (X, d) and (X', d') are two differential A-modules, a homomorphism $f : (X, d) \to (X', d')$ is a homomorphism of A-modules that makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "d"'] \arrow[d, "f"'] & X \arrow[d, "f"'] \\ X' \arrow[r, "d'"'] & X' \end{tikzcd} .$$

Let X be a differential A-module. We set $Z(X) = \mathrm{Ker}(d), B(X) = \mathrm{Im}(d)$ since $B(X) \subseteq Z(X)$, we can define the quotient module

$$H(X) = Z(X)/B(X).$$

Suppose $f : (X, d) \to (Y, d')$ is a homomorphism of differential A-modules. If $x \in Z(X)$ then $f(d(x)) = 0 = d'(f(x))$ so $f(x) \in Z(X)$. Similarly, f maps $B(X)$ into $B(Y)$. Thus we get a map $f^* : H(X) \to H(Y)$ that is an A-module homomorphism. The preceding discussion shows us that we can consider $X \mapsto H(X)$ as a covariant functor from the category of differential A-modules to the category of A-modules.

2. Complexes

Let A be a ring, we call a sequence of A-modules $X = (X_n)_{n\in\mathbb{Z}}$ a *graded* A*-module*. The component X_n is called the *degree* n part of X. Let X and Y be two graded A-modules, by a *homomorphism of degree* r from X to Y we mean a sequence of A-module homomorphisms $f_n : X_n \to Y_{n+r}$; if $r = 0$ then we say that f is a *homomorphism* from X to Y.

A *complex* over a base ring A is a graded A-module X together with a homomorphism $d : X \to X$ of a certain degree r that satisfies $d \circ d = 0$. The following two cases are important:

(1) The *complex of chains*; here we suppose $X_n = 0$ for $n < 0$ and d is of degree -1. The X_n are called *chains* of dimension n, d is called the *boundary operator* of X; an $x \in X$ is called a *cycle* if $dx = 0$, and a *boundary* if there is an $x' \in X_{n+1}$ such that $x = dx'$. In X_n the cycles form a submodule $Z_n(X)$, the boundaries form a submodule $B_n(X)$, and we call the factor group $H_n(X) = Z_n(X)/B_n(X)$ the n-th *homology group*.

(2) The *complex of cochains*; here we again suppose $X_n = 0$ for $n < 0$ and d is of degree $+1$. We write X^n for X_n in this case. We call the chains, boundaries and boundary map in this situation cochains, coboundaries and the coboundary map respectively. We set $Z^n(X) = \mathrm{Ker}(X^n \to X^{n+1})$, $B^n(X) = \mathrm{Im}(X^{n-1} \to X^n)$ and define $H^n(X) = Z^n(X)/B^n(X)$ to be the n-th *cohomology group*. If X and Y are chain (resp. cochain) complexes and $f : X \to Y$ is a map of complexes, then it induces maps on the homology (resp. cohomology) groups which we call f_* (resp. f^*).

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3. Homotopy

If X and Y are two differential A-modules, and $f_0, f_1 : X \to Y$ two maps of differential A-modules, we say that f_0 and f_1 are *homotopic* if there exists an A-linear map $h : X \to Y$ such that $f_1 - f_0 = h \circ d + d' \circ h$. Suppose $dx = 0$ so that $x \in Z(X)$, then $f_1(x) - f_0(x) = d(h(x))$; consequently f_1 and f_0 both induce the same map $f_* : H(X) \to H(Y)$. The relation of homotopy is an equivalence relation.

We say that two differential A-modules X and Y are *homotopy equivalent* if there exist homomorphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity; $H(X)$ and $H(Y)$ are isomorphic in this case. In particular we say that a differential A-module X is *homotopically trival* if the endomorphisms 0 and 1 of X are homotopic, in this case $H(X) = 0$ however, the converse is not true.

Theorem 3.1. *A differential A-module X is homotopically trivial iff one of the following two conditions are satisfied:*

(1) $H(X) = 0$ and $Z(X)$ is a direct factor of X;
(2) for all A-modules L, the differential group $\hom_A(X, L)$ is acyclic, i.e., $H(\hom_A(X, L)) = 0$.

Proof : Suppose that X is homotopically trivial. Then in $H(X)$ we have $x = 1(x) = 0(x) = 0$, and thus $H(X) = 0$. We argue that $\hom(X, L)$ is homotopically trivial and hence acyclic. Now $\mathbf{1}(f) = f$, but X is homotopically trivial so $f(x) - f(0) = f(h \circ d)(x) + f(d \circ h)(x)$. Define $H : \hom(X, L) \to \hom(X, L)$ by $f \mapsto f \circ h$, and recalling that the derivation on $\hom(X, L)$ is $D : f \mapsto f \circ d$, we see that $f(x) - f(0) = (D(H)(f) - H(D)(f))(x)$. But $f(0) = \mathbf{0}(f)$ so $\hom(X, L)$ is homotopically trivial. We need to argue that $Z(X)$ is a direct factor of X. Let $u \in \hom(X, L)$ be such that $D(u) = 0$ this says that $u(d(x)) = 0$ or that u is 0 on $B(X) = Z(X)$. Since $\hom(X, L)$ is acyclic we also get that there is a $v \in \hom(X, L)$ such that $u = D(v) = v \circ d$. This means that the homomorphism $v : Z(X) \to L$ defined by $v(d(x)) = u(x)$ extends to X. Now by the same process we can extend every map $v : Z(X) \to L$ to a map $u = v \circ d : X \to L$. Thus every map from $Z(X)$ to L extends to X; this shows that $Z(X)$ is a direct factor X.

Suppose conversely that (1) is satisfied, then there is a submodule $M \subseteq X$ such that each $x \in X$, can be written in the form $x = m + dy$ and also of the form $m + dm'$ for $m, m' \in M$ uniquely determined by x; setting $m' = h(x)$ we find that $(h \circ d + d \circ h)(x) = (h \circ d + d \circ h)(m + dm') = h \circ dm + d \circ h(dm') = m + dm' = 1(x)$.

Now suppose (2) is satisfied, let $v \in \hom(X, L)$ satisfy $v(B(X)) = 0$. Then v is of the form $v = w \circ d$ for some $w : X \to L$. Each such v is of course zero on $Z(X)$. This implies that $B(X) = Z(X)$. Furthermore, we have seen that any map $v : Z(X) \to L$ extends to a map of X by $v \circ d : X \to L$ and so $Z(X)$ is a direct factor X. $\square$

4. The exterior algebra complex

Let A be a ring, $\mathbf{f} = (f_i)_{1 \le i \le r}$ a system of r elements of A. The exterior algebra complex $\mathbf{K}.(\mathbf{f})$ is chain complex defined as follows: the graded A-module $\mathbf{K}.(\mathbf{f})$ is taken to be the exterior algebra $\wedge A^r$, and the boundary operation is the *interior multiplication* $i_\mathbf{f}$ by $\mathbf{f}$ considered as an element of the dual $(\check{A^r})$; recall that $i_\mathbf{f}$ (antiderivation of degree -1) operates on $\wedge A^r$ by $i_\mathbf{f} \mathbf{e}_i = f_i$ where $(\mathbf{e}_i)_{1 \le i \le r}$ is the canonical basis for $\wedge^r A$. It is clear that $i_\mathbf{f} \circ i_\mathbf{f} = 0$.

For an A-module M, define the chain complex $\mathbf{K}.(\mathbf{f}, M) = \mathbf{K}.(\mathbf{f}) \otimes_A M$ and the cochain complex to be $\mathbf{K}^{\cdot}(\mathbf{f}) = \hom_A(\mathbf{K}.(\mathbf{f}), M)$.

If g is a k-cochain, and setting $g(i_1, \cdots, i_k) = g(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k})$, unwinding definitions we find that

$$d^k g(i_1, \cdots, i_{k+1}) = \sum_{1 \le h \le k+1} (-1)^{h-1} f_{i_h} g(i_1, \cdots, \hat{i}_h, \cdots, i_{k+1}).$$

From these complexes, we define the homology and cohomology groups as usual and define $H_{\cdot}(\mathbf{f}, M) = H_{\cdot}(\mathbf{K}_{\cdot}(\mathbf{f}, M))$ and $H^{\cdot}(\mathbf{f}, M) = H^{\cdot}(\mathbf{K}^{\cdot}(\mathbf{f}, M))$.
We can define a A-isomorphism $K_{\cdot}(\mathbf{f}, M) \cong K^{\cdot}(\mathbf{f}, M)$ by sending each chain $z = \sum(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}) \otimes z_{i_1,\cdots i_k}$ the cochain g_z such that $g(j_1, \cdots, j_{r-k}) = \epsilon z_{i_1,\cdots,i_k}$ where $(j_h)_{1 \leq h \leq r-k}$ is the increasing complement sequence to $(i_h)_{1 \leq h \leq k}$ and ϵ is the sign of the permutation $i_1, \cdots, i_k, j_1, \cdots, j_{r-k}$. One verifies that $g_{dz} = d(g_z)$ and this gives an isomorphism

$$H^i(\mathbf{f}, M) \cong H_{r-i}(\mathbf{f}, M) \text{ for } 0 \leq i \leq r.$$