ASYMPTOTIC THEORY FOR ESTIMATING THE SINGULAR VECTORS AND VALUES OF A PARTIALLY-OBSERVED LOW RANK MATRIX WITH NOISE

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Matrix completion algorithms recover a low rank matrix from a small fraction of the entries, each entry contaminated with additive errors. In practice, the singular vectors and singular values of the low rank matrix play a pivotal role for statistical analyses and inferences. This paper proposes estimators of these quantities and studies their asymptotic behavior. Under the setting where the dimensions of the matrix increase to infinity and the probability of observing each entry is identical, Theorem 4.1 gives the rate of convergence for the estimated singular vectors; Theorem 4.3 gives a multivariate central limit theorem for the estimated singular values. Even though the estimators use only a partially observed matrix, they achieve the same rates of convergence as the fully observed case. These estimators combine to form a consistent estimator of the full low rank matrix that is computed with a non-iterative algorithm. In the cases studied in this paper, this estimator achieves the minimax lower bound in Koltchinskii et al. [2011a]. The numerical experiments corroborate our theoretical results.

1. Introduction. The matrix completion problem arises in several different machine learning and engineering applications, ranging from collaborative filtering (Rennie and Srebro [2005]), to computer vision (Weinberger and Saul [2006]), to positioning (Montanari and Oh [2010]), and to recommender systems (Bennett and Lanning [2007]). The literature has established a sizable body of algorithmic research (Rennie and Srebro [2005], Keshavan et al. [2009], Cai et al. [2010], Mazumder et al. [2010], Hastie et al. [2014], Cho et al. [2015]) and theoretical results (Fazel [2002], Srebro et al. [2004], Candès and Recht [2009], Candès and Plan [2010], Keshavan et al. [2010], Recht [2011], Gross [2011], Negahban et al. [2011], Koltchinskii et al. [2011], Cai and Zhou [2013], Davenport et al. [2014], Chatterjee [2014]). This extant literature is primarily focused

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on estimating the unobserved entries of the matrix. In several of these previous estimation techniques, the algorithms first estimate the singular vectors and singular values of the low rank matrix. Based upon classical multivariate statistics, these singular vectors and singular values can serve various types of statistical analyses and inferences. For example, the overarching aim in the Netflix problem was to predict the unobserved film ratings and the previous algorithms and theories served this purpose. However, if one wishes to interpret the resulting model predictions, then the estimated singular vectors and singular values can determine (i) the main latent factors of film preferences and (ii) their relative strengths. In the Netflix example,

“The first factor has on one side lowbrow comedies and horror movies, aimed at a male or adolescent audience (Half Baked, Freddy vs. Jason), while the other side contains drama or comedy with serious undertones and strong female leads (Sophie’s Choice, Moonstruck). The second factor has independent, critically acclaimed, quirky films (Punch-Drunk Love, I Heart Huckabees) on one side, and mainstream formulaic films (Armageddon, Runaway Bride) on the other side.” (Koren et al. [2009])

This inference is based upon the leading singular vectors of the estimated matrix. To the best of our knowledge, no previous research has studied the statistical properties of the estimated singular vectors and singular values.

This paper proposes estimators of the singular vectors and singular values of the low rank matrix as well as an estimator of the low rank matrix itself. First, Lemma 3.1 studies the singular vectors and singular values of a partially observed matrix that simply substitutes zeros for the unobserved entries; the resulting estimators are biased. The proposed estimators adjust for this bias. Theorem 4.1 finds the convergence rate for the bias-adjusted singular vector estimators and Theorem 4.3 gives a multivariate central limit theorem for the bias-adjusted singular value estimators. Despite the fact that the proposed estimators are built upon a partially observed matrix, they converge at the same rate as the standard estimators built from a fully observed matrix up to a constant factor which depends on the probability of observing each entry. Combining the proposed singular vector and value estimators, Section 4.2 gives a one-step consistent estimator of the full matrix.

The rest of this paper is organized as follows. Section 2 describes the model setup. Section 3 shows that the singular vectors and singular values of a partially observed matrix are biased and suggests a bias-adjusted alternative. Section 4.1 finds (1) the convergence rates of the estimated singular vectors and (2) the asymptotic distribution of the estimated singular values. Section 4.2 proposes and studies a one-step consistent estimator of the full matrix.
Section 5 corroborates the theoretical findings with numerical experiments. Finally, Section 6 provides the proofs of our main theoretical results. The proofs of the other results are collected in the Appendix.

2. **Model setup.** The underlying matrix that we wish to estimate is an $n \times d$ matrix $M_0$ with rank $r$. By singular value decomposition (SVD),

\[
M_0 = U \Lambda V^T,
\]

for orthonormal matrices $U = (U_1, \ldots, U_r) \in \mathbb{R}^{n \times r}$ and $V = (V_1, \ldots, V_r) \in \mathbb{R}^{d \times r}$ containing the left and right singular vectors, and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^{r \times r}$ containing the singular values. $M_0$ is corrupted by noise $\epsilon \in \mathbb{R}^{n \times d}$, where the entries of $\epsilon$ are i.i.d. sub-Gaussian random variables with mean zero and variance $\sigma^2$. Let $y \in \{0, 1\}^{n \times d}$ be such that $y_{ij} = 1$ if the $(i, j)$-th entry of $M_0 + \epsilon$ is observed and $y_{ij} = 0$ if it is not observed. The entries of $y$ are i.i.d. Bernoulli($p$) and independent of the entries of $\epsilon$. We observe $y$ and the partially observed matrix $M \in \mathbb{R}^{n \times d}$,

\[
M_{ij} = [y \cdot (M_0 + \epsilon)]_{ij} = \begin{cases} M_{0ij} + \epsilon_{ij} & \text{if observed } (y_{ij} = 1) \\ 0 & \text{otherwise } (y_{ij} = 0) \end{cases}
\]

for $1 \leq i \leq n$ and $1 \leq j \leq d$. Throughout the paper, it is presumed that $r \ll d \leq n$. Moreover, the entries of $M_0$ are bounded in absolute value by a constant $L$.

3. **Estimation of singular values and vectors of $M_0$.** The vast majority of previous estimators of $M_0$ have been initialized with $M$, in effect imputing the missing values with zero. In this section, we study the properties of singular vectors and values of $M$. This suggests alternative estimators of the singular vectors and values of $M_0$.

3.1. **Properties of singular values and vectors of $M$.** Define

\[
\hat{\Sigma} := M^T M \quad \text{and} \quad \hat{\Sigma}_t := MM^T.
\]

Then, the eigenvectors of $\hat{\Sigma}$ and $\hat{\Sigma}_t$ are the same as the right and left singular vectors of $M$, respectively, and the squared root of eigenvalues of $\hat{\Sigma}$ are the same as the singular values of $M$. The following lemma shows that $\hat{\Sigma}$ and $\hat{\Sigma}_t$ are biased estimators of $M_0^T M_0$ and $M_0 M_0^T$, respectively.

**Lemma 3.1.** Under the model setup in Section 2, we have

\[
\mathbb{E} \hat{\Sigma} = p^2 M_0^T M_0 + p(1 - p) \text{diag}(M_0^T M_0) + np\sigma^2 I_d,
\]
and similarly,
\begin{equation}
\mathbb{E} \hat{\Sigma}_t = p^2 M_0 M_0^T + p(1 - p) \text{diag}(M_0 M_0^T) + dp\sigma^2 I_n,
\end{equation}
where $I_d$ and $I_n$ are $d \times d$ and $n \times n$ identity matrices, respectively.

The proof of this lemma is in Appendix A.1. The right-hand side of (2) contains terms beyond $p^2 M_0^T M_0$ and they make the singular vectors and singular values of $M_0$ biased estimators of the singular vectors and values of $M_0$. While the bias coming from $np\sigma^2 I_d$ is manageable,\(^1\) the bias coming from $p(1 - p) \text{diag}(M_0^T M_0)$ is not. The same applies to $\hat{\Sigma}_t$ in (3).

To get rid of the terms producing unmanageable biases, we define $\hat{\Sigma}_p$ and $\hat{\Sigma}_{pt}$ and their eigenvectors and eigenvalues as follows,
\begin{equation}
\begin{align*}
\hat{\Sigma}_p &:= \hat{\Sigma} - (1 - p) \text{diag}(\hat{\Sigma}) \\
&= (V_p, V_{pc}) \text{diag}(\lambda_{p1}^2, \ldots, \lambda_{pd}^2) (V_p, V_{pc})^T, \quad \text{and} \\
\hat{\Sigma}_{pt} &:= \hat{\Sigma}_t - (1 - p) \text{diag}(\hat{\Sigma}_t) \\
&= (U_p, U_{pc}) \text{diag}(\lambda_{pt1}^2, \ldots, \lambda_{ptn}^2) (U_p, U_{pc})^T,
\end{align*}
\end{equation}
where
\begin{align*}
V_p &= (V_{p1}, \ldots, V_{pr}) \in \mathbb{R}^{d \times r}, & V_{pc} &= (V_{pr+1}, \ldots, V_{pd}) \in \mathbb{R}^{d \times (d - r)}, \\
U_p &= (U_{p1}, \ldots, U_{pr}) \in \mathbb{R}^{n \times r}, & U_{pc} &= (U_{pr+1}, \ldots, U_{pn}) \in \mathbb{R}^{n \times (n - r)}.
\end{align*}

The following proposition shows that $\hat{\Sigma}_p$ and $\hat{\Sigma}_{pt}$ adjust the bias.

**Proposition 3.1.** Under the model setup in Section 2, we have by eigen-decomposition,
\begin{align*}
\mathbb{E} \hat{\Sigma}_p &= p^2 M_0^T M_0 + np^2 \sigma^2 I_d = (V, V_c) \hat{\Lambda}_p^2 (V, V_c)^T \quad \text{and} \\
\mathbb{E} \hat{\Sigma}_{pt} &= p^2 M_0^T M_0 + dp^2 \sigma^2 I_n = (U, U_c) \hat{\Lambda}_{pt}^2 (U, U_c)^T,
\end{align*}
where $V$ and $U$ are as defined in (1), $V_c \in \mathbb{R}^{d \times (d - r)}$, $U_c \in \mathbb{R}^{n \times (n - r)}$,
\begin{align*}
\hat{\Lambda}_p^2 &= \text{diag}(\hat{\lambda}_{p1}^2, \ldots, \hat{\lambda}_{pd}^2) \\
&= \text{diag}(p^2[\lambda_1^2 + n\sigma^2], \ldots, p^2[\lambda_r^2 + n\sigma^2], p^2n\sigma^2, \ldots, p^2n\sigma^2) \in \mathbb{R}^{d \times d}, \quad \text{and} \\
\hat{\Lambda}_{pt}^2 &= \text{diag}(p^2[\lambda_{pt1}^2 + d\sigma^2], \ldots, p^2[\lambda_{ptn}^2 + d\sigma^2], p^2d\sigma^2, \ldots, p^2d\sigma^2) \in \mathbb{R}^{n \times n}.
\end{align*}

\(^1\)This term does not change the singular vectors of $\mathbb{E} \hat{\Sigma}_t$; it merely increases each singular value by $np\sigma^2$.\[\]
The proof of this proposition easily follows from Lemma 3.1 and (4).

Proposition 3.1 shows that the top \( r \) eigenvectors of \( \Sigma p \) and \( \Sigma p t \) are the same as the right and left singular vectors of \( M_0 \), respectively. Also, the top \( r \) eigenvalues of \( \Sigma p \) are easily adjusted to match the singular values of \( M_0 \) as follows,

\[
\lambda_i^2 = \frac{1}{p^2} \hat{\lambda}_p^2 - n \sigma^2, \quad \text{for } i = 1, \ldots, r.
\]

3.2. Estimators of singular values and vectors of \( M_0 \). The results in Proposition 3.1 suggest plug-in estimators using the leading eigenvectors and eigenvalues of \( \hat{\Sigma} p \) and the leading eigenvectors of \( \hat{\Sigma} pt \) as estimators of \( V, \Lambda, \) and \( U \), respectively. However, since \( p \) is an unknown parameter in practice, the proposed estimators use \( \hat{p} \) to estimate \( p \).

\[
\hat{p} = \frac{\sum_{k=1}^n \sum_{h=1}^d y_{kh}}{nd}.
\]

Using \( \hat{p} \), define \( \hat{\Sigma} \) and \( \hat{\Sigma} pt \) as

\[
\hat{\Sigma} = \hat{\Sigma} - (1 - \hat{p}) \text{diag}(\hat{\Sigma}) \quad \text{and} \quad \hat{\Sigma} pt = \hat{\Sigma} - (1 - \hat{p}) \text{diag}(\hat{\Sigma} t).
\]

By eigendecomposition,

\[
\hat{\Sigma} = (\hat{V}, \hat{V}_c) \Lambda_\hat{p}^2 (\hat{V}, \hat{V}_c)^T \quad \text{and} \quad \hat{\Sigma} pt = (\hat{U}, \hat{U}_c) \Lambda^2 pt (\hat{U}, \hat{U}_c)^T,
\]

where \( \hat{V} \in \mathbb{R}^{d \times r} \), \( \hat{V}_c \in \mathbb{R}^{d \times (d-r)} \), \( \Lambda_\hat{p}^2 = \text{diag}(\lambda_{\hat{p}1}^2, \ldots, \lambda_{\hat{p}d}^2) \in \mathbb{R}^{d \times d} \), \( \hat{U} \in \mathbb{R}^{n \times r} \), \( \hat{U}_c \in \mathbb{R}^{n \times (n-r)} \), and \( \Lambda^2 pt = \text{diag}(\lambda_{pt1}^2, \ldots, \lambda_{ptn}^2) \in \mathbb{R}^{n \times n} \). Then, estimate the left and right singular vectors, \( U \) and \( V \), of \( M_0 \) by \( \hat{U} \) and \( \hat{V} \), respectively. Also, estimate the singular values, \( \lambda_i, i = 1, \ldots, r \), of \( M_0 \) by

\[
\hat{\lambda}_i = \sqrt{\frac{1}{p^2} (\lambda_{p\hat{p}}^2 - \hat{\tau}_p)} \quad \text{for } i = 1, \ldots, r,
\]

where \( \hat{\tau}_p = \frac{1}{d-r} \text{tr} \left( \hat{V}^T c \hat{\Sigma} p \hat{V}_c \right) \).

For any \( A \in \mathbb{R}^{n \times d} \), let the \( i \)-th left singular vector of \( A \) be denoted by \( u_i(A) \), the \( i \)-th right singular vector of \( A \) by \( v_i(A) \), and the top \( i \)-th singular value of \( A \) by \( \lambda_i(A) \) for \( i = 1, \ldots, d \). Then, Algorithm 1 summarizes the steps to compute the proposed estimators of the singular values and vectors of \( M_0 \).

4. Asymptotic theory. This section investigates the statistical properties of the estimators proposed in (7) and (8).
The squared Frobenius norm is defined by
\[ \|A\|_F^2 = \text{tr} (A^T A), \]
and the maximum singular value is defined as follows,
\[ \lambda_i = \max_{1 \leq i \leq d} \sum_{j=1}^d |A_{ij}|, \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^d |A_{ij}|. \]

Algorithm 1

\begin{algorithm}
\caption{Estimators of $U_i$, $V_i$, and $\lambda_i$ for $i = 1, \ldots, r$}
\begin{algorithmic}
\Require $M$, $y$, and $r$
\State $\hat{p} \leftarrow \frac{1}{nd} \sum_{k=1}^n \sum_{h=1}^d y_{kh}$
\State $\hat{\Sigma}_p \leftarrow M^T M - (1 - \hat{p}) \text{diag}(M^T M)$
\State $\hat{\Sigma}_{p,l} \leftarrow MM^T - (1 - \hat{p}) \text{diag}(MM^T)$
\State $\hat{V}_i \leftarrow u_i(\hat{\Sigma}_p), \quad \forall i \in \{1, \ldots, r\}$
\State $\hat{U}_i \leftarrow u_i(\hat{\Sigma}_{p,l}), \quad \forall i \in \{1, \ldots, r\}$
\State $\hat{\lambda}_i \leftarrow \frac{1}{d-r} \sum_{i=r+1}^d \lambda_i(\hat{\Sigma}_p)$
\State \Return $\hat{V}_i, \hat{U}_i$, and $\hat{\lambda}_i$ for $i = 1, \ldots, r$
\end{algorithmic}
\end{algorithm}

4.1. Convergence rate of the estimated singular vectors and asymptotic distribution of the estimated singular values. Let $x = (x_1, \ldots, x_n)^T$ be a $n$-dimensional vector and $A = (A_{ij})$ a $n \times d$ matrix. Then, the $\ell_p$-norm is defined as follows,
\[ \|x\|_p = \left( \sum_{i=1}^p |x_i|^p \right)^{1/p}, \quad \text{and} \quad \|A\|_p = \sup \{\|Ax\|_p : \|x\|_p = 1\}, \quad p = 1, 2, \infty. \]
The spectral norm $\|A\|_2$ is a square root of the largest eigenvalue of $AA^T$,
\[ \|A\|_1 = \max_{1 \leq j \leq d} \sum_{i=1}^n |A_{ij}|, \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^d |A_{ij}|. \]
The squared Frobenius norm is defined by $\|A\|_F^2 = \text{tr} (A^T A)$, the trace of $A^T A$. We denote by $c > 0$ and $C > 0$ generic constants that are free of $n$, $d$, and $p$, and different from appearance to appearance.

To measure how close the proposed estimator $\hat{V}$ is to $V$ (or, $\hat{U}$ to $U$), we introduce a classical notion of distance between subspaces. Let $\mathcal{R}(Z_1)$ denote a column space spanned by $Z_1 \in \mathbb{R}^{d \times r}$ and $\mathcal{R}(Z_2)$ by $Z_2 \in \mathbb{R}^{d \times r}$. Then, to measure the dissimilarity between $\mathcal{R}(Z_1)$ and $\mathcal{R}(Z_2)$, consider the following loss function
\[ \|\sin(Z_1, Z_2)\|_F^2 = \|\sin(\Theta(\mathcal{R}(Z_1), \mathcal{R}(Z_2)))\|_F^2, \]
where $\sin(\Theta(\mathcal{R}(Z_1), \mathcal{R}(Z_2)))$ is a diagonal matrix of singular values (canonical angles) of $P_1 P_2^T$ with orthogonal projections $P_1$ and $P_2$ of $Z_1$ and $Z_2$, respectively. Here $P_{1}^\perp = I - P$. The canonical angles generalize the notion of angles between lines and are often used to define the distance between subspaces. If the columns of $Z_1$ and $Z_2$ are singular vectors, $\mathcal{R}(Z_1)$ and $\mathcal{R}(Z_2)$ have projections $P_1 = Z_1 Z_1^T$ and $P_2 = Z_2 Z_2^T$, respectively, and
\[ \| \sin(Z_1, \tilde{Z}_2) \|_F^2 = \| Z_1 Z_1^T (Z_2 Z_2^T)^{-1} \|_F^2 = \frac{1}{2} \| Z_1 Z_1^T - Z_2 Z_2^T \|_F^2. \]

Proposition 2.2 in Vu and Lei [2013] relates this subspace distance to the Frobenius distance

\[ \frac{1}{2} \inf_{O \in \mathbb{V}_{r,r}} \| Z_1 - Z_2 O \|_F^2 \leq \| \sin(Z_1, Z_2) \|_F^2 \leq \inf_{O \in \mathbb{V}_{r,r}} \| Z_1 - Z_2 O \|_F^2, \]

where \( \mathbb{V}_{r,r} = \{ O \in \mathbb{R}^{r \times r} : O^T O = I_r \text{ and } O O^T = I_r \} \) denotes the Stiefel manifold of \( r \times r \) orthonormal matrices. In other words, the distance between two subspaces corresponds to the minimal distance between their orthonormal bases.

**Assumption 1.**

1. \( \lambda_k = b_k \sqrt{nd}, k = 1, \ldots, r, \) where \( \frac{1}{c} \leq b_k \leq c \) for a constant \( c > 0; \)
2. there exists a constant \( m \in \{1, \ldots, r\} \) such that \( b_m > b_{m+1} \), where \( b_{r+1} = 0; \)
3. \( d \leq n \leq c^{dn} \) for a constant \( \alpha < 1 \) free of \( n, d, \) and \( p. \)

**Remark 1.** To motivate Assumption 1 (1), suppose that a non-vanishing proportion of entries of \( M_0 \) contains non-vanishing signals (i.e. \( M_0^2 \geq c_0 \) for some constant \( c_0 > 0 \)) and that the rank of \( M_0 \) is fixed. Then,

\[ \sum_{i=1}^{n} \sum_{j=1}^{d} M_{0ij}^2 = \|M_0\|_F^2 \geq cnd \]

for some constant \( c > 0. \) Because the squared Frobenius norm is also the sum of the squared singular values of \( M_0, \) the order of the singular values of \( M_0 \) should be \( \sqrt{nd} \) (see also Fan et al. [2013]). Assumption 1 (1) may seem uncommon in the matrix completion literature, but the widely-used assumption (II.2) in Candès and Plan [2010] implies Assumption 1(1).

The following theorem shows the convergence of \( \hat{V} \) to \( V \) and \( \hat{U} \) to \( U. \)

**Theorem 4.1.** Under the model setup in Section 2 and Assumption 1, let \( \hat{V}^{(m)} \) and \( \hat{U}^{(m)} \) be the first \( m \) columns of \( V \) and \( U \) defined in (7), respectively, and let \( V^{(m)} \) and \( U^{(m)} \) be the first \( m \) columns of \( V \) and \( U \) defined in (1), respectively. Then, for large \( n \) and \( d, \)

\[ \mathbb{E} \left\| \sin(\hat{V}^{(m)}, V^{(m)}) \right\|_F^2 \leq \frac{C_1 n^{-1}}{p (b_m^2 - b_{m+1}^2)^2} \]
and

\[
E\left\| \sin (\hat{U}^{(m)}, U^{(m)}) \right\|^2_F \leq \frac{C_2 d^{-1}}{p(b_m^2 - b_{m+1}^2)^2},
\]

where \(C_1\) and \(C_2\) are generic constants free of \(n, d,\) and \(p.\)

The proof of this theorem is in Section 6.1.

**Remark 2.** As long as \(pd/\log n \to \infty,\) the convergence results in Theorem 4.1 will hold.

**Remark 3.** Despite the fact that \(\hat{V}^{(m)}\) is built on a partially observed matrix \(M,\) Theorem 4.1 gives the convergence rate \(n^{-1/2}\) which is the standard convergence rate for eigenvectors (Anderson et al. [1958]). The effect of the partial observations appears in the denominator of the right-hand side of (10) as \(p.\) A similar story applies to \(\hat{U}^{(m)}\) in (11).

The next theorem shows the asymptotic distribution of \(\hat{\lambda}_i^2\) centered around \(\lambda_i^2.\)

**Theorem 4.2.** Suppose \(nd^{-1} \to \infty.\) Then, under the model setup in Section 2 and Assumption 1, we have

\[
\frac{\sum_{i=1}^{m} \hat{\lambda}_i^2 - \sum_{i=1}^{m} \lambda_i^2}{\sqrt{nd}\sigma_\lambda} \to N(0, 1) \text{ in distribution, as } n \text{ and } d \to \infty.
\]

where

\[
\sigma_\lambda^2 = \frac{4(1-p)}{p} \left\{ \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh} \left( \sum_{i=1}^{m} b_i U_{ik} V_{ih} \right)^2 - \left( \sum_{i=1}^{m} b_i^2 \right)^2 \right\} + \frac{4\sigma^2}{p} \sum_{i=1}^{m} b_i^2,
\]

\(U_{ik}\) is the \(k\)-th entry of \(U_i,\) and \(V_{ih}\) is the \(h\)-th entry of \(V_i.\)

The proof of this theorem is in Section 6.2.

**Remark 4.** As long as \(pd/\log n \to \infty\) and \(pnd^{-1} \to \infty,\) the asymptotic normality result in Theorem 4.2 will hold.

**Remark 5.** Theorem 4.2 shows that the convergence rate of \(\sum_{i=1}^{m} \hat{\lambda}_i^2\) is \(\sqrt{nd}.\) Considering Assumption 1(1), it is an optimal rate. However, since the results are based on partially observed entries, the asymptotic variance, \(\sigma_\lambda^2,\) increases with the rate \(p^{-1}.\) For example, when we have a fully-observed matrix, \(\sigma_\lambda^2\) simply becomes \(4\sigma^2 \sum_{i=1}^{m} b_i^2\) which is a lower bound for \(\sigma_\lambda^2.\)
One of the main purposes of this paper is to investigate asymptotic behaviors of the estimators of the singular values of $M_0$. An application of the proof of Theorem 4.2 and the delta method provides a multivariate central limit theorem for $\hat{\lambda}_1, \ldots, \hat{\lambda}_r$.

**Theorem 4.3.** Suppose that

$$b_i > b_{i+1} \quad \text{for all} \quad i \in \{1, \ldots, r\} \quad \text{and} \quad nd^{-1} \to \infty.$$ 

Then, under the model setup in Section 2 and Assumption 1, we have

$$\Upsilon^{-1/2} \begin{pmatrix} \hat{\lambda}_1 - \lambda_1 \\ \vdots \\ \hat{\lambda}_r - \lambda_r \end{pmatrix} \to \mathcal{N}(0, I_r) \quad \text{in distribution, as} \quad n \text{ and } d \to \infty,$$

where $\Upsilon = \Upsilon^T \in \mathbb{R}^{r \times r}$ consists of

$$\Upsilon_{ij} = \begin{cases} \frac{(1-p)}{p} \left( \sum_{k=1}^n \sum_{h=1}^d M_{0kh}^2 U_{ik}^2 V_{ih}^2 - b_i^2 \right) + \sigma_i^2 & \text{if } i = j \\ \frac{(1-p)}{p} \left( \sum_{k=1}^n \sum_{h=1}^d M_{0kh}^2 U_{ik} V_{ih} U_{jk} V_{jh} - b_i b_j \right) & \text{if } i \neq j. \end{cases}$$

Thus, $|\hat{\lambda}_i - \lambda_i| = O_p \left( \frac{1}{\sqrt{n}} \right)$.

Theorems 4.1-4.3 show that the proposed estimators for $U, V$, and $\lambda_i$'s are asymptotically unbiased and have optimal convergence rates. With these well-developed estimators for the singular values and vectors of $M_0$, the following section proposes a consistent estimator of $M_0$.

4.2. A consistent estimator of $M_0$. Suppose that $b_i > b_{i+1}$ for all $i = 1, \ldots, r$. Theorem 4.1 and (9) imply that $\hat{V}_i$ and $\hat{U}_i$ can estimate $V_i$ and $U_i$ up to constant factors $\text{sign}((\hat{V}_i, V_i))$ and $\text{sign}((\hat{U}_i, U_i))$, respectively. Let $s_0 = (s_{01}, \ldots, s_{0r}) \in \{-1, 1\}^r$ be

$$s_{0i} = \text{sign}((\hat{V}_i, V_i)) \text{sign}((\hat{U}_i, U_i)) \quad \text{for} \quad i \in \{1, \ldots, r\}. \quad (12)$$

Then, $\hat{M}(s_0) = \sum_{i=1}^r s_{0i} \hat{\lambda}_i \hat{U}_i \hat{V}_i^T$ becomes a consistent estimator of $M_0$. However, since $s_0$ is an unknown parameter in practice, we employ $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_r) \in \{-1, 1\}^r$ as an estimator of $s_0$;

$$\hat{s} = \arg\min_{s \in \{-1, 1\}^r} \| \mathcal{P}_\Omega(\hat{M}(s)) - \mathcal{P}_\Omega(M) \|_F^2.$$
where $\Omega$ contains indices of the observed entries, $y_{ij} = 1 \iff (i, j) \in \Omega$, and $\mathcal{P}_\Omega(A)$ for any $A \in \mathbb{R}^{n \times d}$ denotes the projection of $A$ onto $\Omega$,

$$
\mathcal{P}_\Omega(A)_{ij} = \begin{cases} 
A_{ij} & \text{if } (i, j) \in \Omega \\
0 & \text{if } (i, j) \notin \Omega
\end{cases}
$$

for $1 \leq i \leq n$ and $1 \leq j \leq d$.

Hence, the proposed estimator of $M_0$ is

$$
\hat{M}(\hat{s}) = \sum_{i=1}^{r} \hat{s}_i \hat{\lambda}_i \hat{U}_i \hat{V}_i^T.
$$

In the following we show that $\hat{M}(\hat{s})$ is a consistent estimator of $M_0$ under certain conditions. The steps to compute $\hat{M}(\hat{s})$ using $\{\hat{V}_i, \hat{U}_i, \hat{\lambda}_i\}_{i=1}^{r}$ from Algorithm 1 are summarized in Algorithm 2.

**Algorithm 2** Estimator of $M_0$

Require: $\hat{V}_i, \hat{U}_i,$ and $\hat{\lambda}_i$ for $i = 1, \ldots, r$

1. $\hat{s} \leftarrow \arg \min_{s \in \{-1, 1\}^r} \| \mathcal{P}_\Omega(\sum_{i=1}^{r} s_i \hat{\lambda}_i \hat{U}_i \hat{V}_i^T) - \mathcal{P}_\Omega(M) \|_F^2$

2. $\hat{M}(\hat{s}) \leftarrow \sum_{i=1}^{r} \hat{s}_i \hat{\lambda}_i \hat{U}_i \hat{V}_i^T$

return $\hat{M}(\hat{s})$

**Assumption 2.**

1. $\lim_{n \to \infty, d \to \infty} \mathbb{P}\left( \min_{s \in \{-1, 1\}^r} \| \mathcal{P}_\Omega(\hat{M}(s)) - \mathcal{P}_\Omega(M) \|_F^2 < \| \mathcal{P}_\Omega(\hat{M}(s_0)) - \mathcal{P}_\Omega(M) \|_F^2 \right) = 0$;

2. $b_i > b_{i+1}$ for all $i = 1, \ldots, r$.

**Remark 6.** When the rank $r$ is 1, it is more straightforward to understand Assumption 2(1). Assuming that $s_0 = 1$, it means that

$$
\lim_{n \to \infty, d \to \infty} \mathbb{P}\left( \| \mathcal{P}_\Omega(-\hat{\lambda} \hat{U} \hat{V}^T) - \mathcal{P}_\Omega(M) \|_F^2 < \| \mathcal{P}_\Omega(\hat{\lambda} \hat{U} \hat{V}^T) - \mathcal{P}_\Omega(M) \|_F^2 \right) = 0.
$$

That is, the probability that $\hat{s}$ picks a different sign than the true sign $s_0 = 1$ goes to zero with the dimensionality. Given the asymptotic properties of our estimators $\hat{\lambda}, \hat{U},$ and $\hat{V}$, this is not an unreasonable assumption to make.
**Theorem 4.4.** Under the model setup in Section 2 and Assumptions 1-2, for any given \( \eta > 0 \), there exists a constant \( C_\eta > 0 \) such that for sufficiently large \( n \),

\[
\mathbb{P}\left( \frac{pb_4}{n} \| \hat{M}(\hat{s}) - M_0 \|_F^2 \geq C_\eta \right) \leq \eta.
\]

Or alternatively,

\[
\| \hat{M}(\hat{s}) - M_0 \|_F^2 = \frac{1}{pb_4} \rho_p(h_n n),
\]

where \( h_n \) can be anything that diverges very slowly with the dimensionality, for example, \( \log(\log d) \).

The proof of this theorem is in Section 6.3.

**Remark 7.** As long as \( \frac{pd}{\log n} \to \infty \), the convergence results in Theorem 4.4 will hold. The conditions that \( pn^d \to \infty \) and \( \alpha_1 > 1 \) that were needed for Theorem 4.2-4.3 are no longer needed. Even when \( n = d \), the convergence results in Theorem 4.4 hold.

**Remark 8.** Theorem 4.4 shows that \( \frac{1}{nd} \| \hat{M}(\hat{s}) - M_0 \|_F^2 \) is bounded by \( Cp^{-1}d^{-1} \) for some constant \( C > 0 \). Under the setting where the rank of \( M_0 \) is fixed as in this paper, this is matched to the minimax lower bound in Theorems 5-7 (Koltchinskii et al. [2011a]). The previous estimators that obtain the minimax rate are computed via semidefinite programs that require iterating over several SVDs. However, the proposed estimator is a non-iterative algorithm. Also, the only tuning parameter that it needs is the rank of \( M_0 \) which is relatively easy to find.

5. **Numerical experiments.** This section studies the performance of the proposed estimators using several values of the dimension \( n \) and the probability \( p \).

To simulate \( M_0 \), generate \( A \in [-5, 5]^{n \times 2} \), \( B \in [-5, 5]^{d \times 2} \) to contain i.i.d. Uniform\([-5, 5]\) random variables and define

\[ M_0 = AB^T \in \mathbb{R}^{n \times d}. \]

Each entry of \( M_0 \) is observed with probability \( p \) and unobserved with probability \( 1 - p \). The observed entries of \( M_0 \) are corrupted by noise \( \epsilon \) as defined in Section 2, where \( \epsilon_{ij} \) are i.i.d. \( \mathcal{N}(0, 1) \). The dimension \( n \) varies from 100 to 1000 and \( p \) from 0.1 to 1, while \( d = 2\sqrt{n} \). Each simulation was repeated 500 times and the errors were averaged.
Fig 1. The mean squared errors for six different values of $p$ when $n$ increases. Each point on the plots correspond to an average over 500 replicates.
Fig 2. The same mean squared errors as the ones in Figure 1 plotted for four different values of \( n \) when \( p \) increases. Each point on the plots correspond to an average over 500 replicates.
Fig 3. Asymptotic normality of \( \sum_{i=1}^{2} \hat{\lambda}_i - \sum_{i=1}^{2} \lambda_i \) as \( p \) varies from 0.1 to 1. Across the plots, we fixed \( n \) to be 1000.
Figures 1 and 2 summarize the resulting mean squared errors calculated by \( \frac{1}{n} \| M(s) - M_0 \|_F^2 \), \( \| \text{diag}(\hat{\lambda}_1, \hat{\lambda}_2) - \Lambda \|_F^2 \), \( \| V - V_0 \|_F^2 \), and \( \| \hat{U} - U \|_F^2 \), when \( n \) and \( p \) increase along the x-axis, respectively. The MSE for \( \hat{V} \) decreases more rapidly than the MSE for \( \hat{U} \) and both MSEs decrease when \( p \) increases; this is consistent with the results in Theorem 4.1. The MSE of \( \hat{M} \) decreases with the increase of \( n \) and \( p \). The MSE of \( \hat{\lambda} \) stays stable over the changes of \( n \) since it is measured on \( \hat{\lambda}_i \) instead of \( \hat{\lambda}_i^2 \) (see Theorem 4.3), but decreases with the increase of \( p \).

We further studied the asymptotic normality of \( \sum_{i=1}^{p} \hat{\lambda}_i^2 \) in Theorem 4.3. Figure 3 graphs the QQ plot of \( \sum_{i=1}^{p} \hat{\lambda}_i^2 - \sum_{i=1}^{p} \lambda_i^2 \), where the dimension \( n \) is fixed at 1000 and \( p \) varies from 0.1 to 1. This shows that the asymptotic normality holds across various values of \( p \).

6. Proofs.

6.1. Proofs for Theorem 4.1. The proof of the following proposition and lemmas are in Appendix A.2.

**Proposition 6.1.** Under the model setup in Section 2 and Assumption 1, we have for large \( n \) and \( d \),

\[
\mathbb{E} \left\| \sin \left( V_p^{(m)} V^{(m)} \right) \right\|_F^2 \leq \frac{C_1 n^{-1}}{p (b_m^2 - b_{m+1}^2)^2}, \quad \text{and}
\]

\[
\mathbb{E} \left\| \sin \left( U_p^{(m)} U^{(m)} \right) \right\|_F^2 \leq \frac{C_2 d^{-1}}{p (b_m^2 - b_{m+1}^2)^2},
\]

where \( V_p \) and \( U_p \) are defined in (4) and \( C_1 \) and \( C_2 \) are generic constants free of \( n, d, \) and \( p \).

**Lemma 6.1.** Under the model setup in Section 2 and Assumption 1, for any given \( \mu_1 > 0 \), there exists a large constant \( C_{\mu_1} > 0 \) such that

\[
\frac{1}{nd} \left\| \hat{\Sigma}_p - \mathbb{E} \hat{\Sigma}_p \right\|_2 \leq C_{\mu_1} \max \left\{ \frac{\log n}{d}, p^{3/2} \sqrt{\frac{\log n}{n}} \right\}
\]

with probability at least \( 1 - O(n^{-\mu_1}) \), where \( \hat{\Sigma}_p \) is defined in (4). Similarly, for any given \( \mu_2 > 0 \), there exists a large constant \( C_{\mu_2} > 0 \) such that

\[
\frac{1}{nd} \left\| \hat{\Sigma}_{pt} - \mathbb{E} \left( \hat{\Sigma}_{pt} \right) \right\|_2 \leq C_{\mu_2} \max \left\{ \frac{\log n}{d}, p^{3/2} \sqrt{\frac{\log n}{d}} \right\}
\]

with probability at least \( 1 - O(n^{-\mu_2}) \), where \( \hat{\Sigma}_{pt} \) is defined in (4).
Lemma 6.2. Under the model setup in Section 2 and Assumption 1, for any given $\nu_1 > 0$, there exists a large constant $C_{\nu_1} > 0$ such that

$$\frac{1}{nd} \left\| \tilde{\Sigma}_\hat{p} - \hat{\Sigma}_p \right\|_2 \leq C_{\nu_1} \frac{p^{3/2}}{\sqrt{n d / d}}$$

with probability at least $1 - O(n^{-\nu_1})$, where $\tilde{\Sigma}_\hat{p}$ and $\hat{\Sigma}_p$ are defined in (6) and (4), respectively. Similarly, for any given $\nu_2 > 0$, there exists a large constant $C_{\nu_2} > 0$ such that

$$\frac{1}{nd} \left\| \tilde{\Sigma}_{\hat{p}t} - \hat{\Sigma}_{pt} \right\|_2 \leq C_{\nu_2} \frac{p^{3/2}}{\sqrt{n d / d}}$$

with probability at least $1 - O(n^{-\nu_2})$, where $\tilde{\Sigma}_{\hat{p}t}$ and $\hat{\Sigma}_{pt}$ are defined in (6) and (4), respectively.

Lemma 6.3. Under the model setup in Section 2 and Assumption 1, we have for large $n$ and $d$,

$$\mathbb{E} \left\| \frac{1}{nd} \left( \tilde{\Sigma}_\hat{p} - \hat{\Sigma}_p \right) V_p^{(m)} \right\|_F^2 \leq C_1 \max \left\{ \frac{p^3(1-p)}{nd^3}, \frac{p^2(1-p)}{n^2d^{5/2}} \right\}$$

and

$$\mathbb{E} \left\| \frac{1}{nd} \left( \tilde{\Sigma}_{\hat{p}t} - \hat{\Sigma}_{pt} \right) U_p^{(m)} \right\|_F^2 \leq C_2 \max \left\{ \frac{p^3(1-p)}{nd^3}, \frac{p^2(1-p)}{d^2n^{5/2}} \right\},$$

where $\tilde{\Sigma}_\hat{p}$ and $\tilde{\Sigma}_{\hat{p}t}$ are defined in (6), $\hat{\Sigma}_p$, $\hat{\Sigma}_{pt}$, $V_p$, and $U_p$ are defined in (4), and $C_1$ and $C_2$ are generic constants free of $n, d,$ and $p$.

Proof of Theorem 4.1. We only prove (10) because (11) can be proved similarly.

By triangle inequality and Proposition 6.1, we have

$$\mathbb{E}\|\sin(\hat{V}(m), V_p^{(m)})\|_F^2 \leq 4 \mathbb{E}\|\sin(\hat{V}(m), V_p^{(m)})\|_F^2 + 4 \mathbb{E}\|\sin(V_p^{(m)}, V^{(m)})\|_F^2$$

$$\leq 4 \mathbb{E}\|\sin(\hat{V}(m), V_p^{(m)})\|_F^2 + \frac{C n^{-1}}{p (b_m^2 - b_{m+1}^2)^2}.$$  

Now, consider $\mathbb{E}\|\sin(\hat{V}(m), V_p^{(m)})\|_F^2$. Let

$$E_1 = \left\{ \max_{1 \leq i \leq d} \frac{1}{nd} |\hat{\lambda}_{p_i}^2 - \tilde{\lambda}_{p_i}^2| < t_1 \right\},$$
where \( t_1 = C_1' \frac{p \log n}{d} + C_1'' p^{3/2} \sqrt{\frac{\log n}{n}}, \) and
\[
E_2 = \left\{ \frac{1}{nd} \left| \lambda_{p+1}^2 - \lambda_{p+1}^2 \right| < t_2 \right\}.
\]

where \( t_2 = C_2 p^{3/2} \sqrt{\frac{\log n}{nd}}. \) Then, by Weyl’s theorem (Li [1998a]), Lemma 6.1, and Lemma 6.2, we have for large constants \( C_1', C_1'', \) and \( C_2, \)
\[
P(E_1^c) \leq P \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_p \right\|_F \geq t_1 \right) = O \left( n^{-4} \right) \text{ and }
P(E_2^c) \leq P \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_p \right\|_F \geq t_2 \right) = O \left( n^{-4} \right).
\]

Thus, for large \( n \) and \( d, \)
\[
E \| \sin \left( \hat{V}^{(m)}, V_p^{(m)} \right) \|_F^2
= E \left\{ \| \sin \left( \hat{V}^{(m)}, V_p^{(m)} \right) \|_F^2 \right\}_E + E \left\{ \| \sin \left( \hat{V}^{(m)}, V_p^{(m)} \right) \|_F^2 \right\}_E \hat{1}_{E_1 \cap E_2}
\leq m \left( E(1_{E_1}) + E(1_{E_2}) \right) + E \left\{ \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_p \right\|_F \right) V_p^{(m)} \|_F \right\}_E \hat{1}_{E_1 \cap E_2}
\leq cn^{-4} + E \left\{ \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_p \right\|_F \right) V_p^{(m)} \|_F \right\}_E \hat{1}_{E_1 \cap E_2}
\leq cn^{-4} + E \left\{ \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_p \right\|_F \right) V_p^{(m)} \|_F \right\}_E \hat{1}_{E_1 \cap E_2}
\leq cn^{-4} + \frac{C(1-p)}{\left( \frac{b_n^2}{b_{m+1}^2} \right)^2} \max \left\{ \frac{1}{p n d^{3}}, \frac{1}{p^2 n^2 d^3/2} \right\}, \tag{19}
\]

where \( 1_E \) is an indicator function of an event \( E, \) the first inequality holds by the fact that \( \| \sin(\hat{V}^{(m)}, V_p^{(m)}) \|_F^2 \leq m \) and Davis-Kahan sin \( \theta \) theorem (Theorem 3.1 in Li [1998b]), and the last inequality is due to Lemma 6.3.

By (18) and (19), the result (10) follows.
6.2. Proofs for Theorem 4.2. The proof of the following propositions are in Appendix A.3.

Proposition 6.2. Under the assumptions in Theorem 4.2, we have

$$\sqrt{nd} \Gamma_{nd}^{-1/2} \left[ \left( \frac{1}{nd} \sum_{i=1}^{m} \lambda_i^2 \right) - \left( \frac{1}{nd} \sum_{i=1}^{m} \left( \lambda_i^2 + n\sigma^2 \right) \right) \right]$$

$$\rightarrow \mathcal{N}(0, I_2) \text{ in distribution, as } n, d \to \infty,$$

where \( \lambda_i \), \( \lambda_i \), and \( \hat{\rho} \) are defined in (4), (1), and (5), respectively, and \( \Gamma_{nd} = \Gamma_{nd}^{T} \in \mathbb{R}^{2 \times 2} \) consists of

\[
(\Gamma_{nd})_{11} = \frac{4(1-p)}{p} \sum_{k=1}^{n} \sum_{h=1}^{d} M_{k0} \left\{ \sum_{i=1}^{m} b_i U_{hk} V_{ih} \right\}^2 + 4\sigma^2 \sum_{i=1}^{m} b_i^2;
\]

\[
(\Gamma_{nd})_{12} = 2p^2 (1-p) \left( \sum_{i=1}^{m} b_i^2 \right)^2,
\]

and \( (\Gamma_{nd})_{22} = p^5 (1-p) \left( \sum_{i=1}^{m} b_i^2 \right)^2 \).

Proposition 6.3. Under the model setup in Section 2 and Assumption 1, let

\[
\hat{\tau}_p = \frac{1}{d - r} \text{tr} \left( V_{pc}^T \hat{\Sigma}_p V_{pc} \right),
\]

where \( \hat{\Sigma}_p \) and \( V_{pc} \) are defined in (4). Then, we have \( \hat{\tau}_p - np^2 \sigma^2 = O_p(p\sqrt{n}) \).

Proof of Theorem 4.2. We have

\[
\frac{1}{\sqrt{nd}} \left\{ \sum_{i=1}^{m} \lambda_i^2 - \sum_{i=1}^{m} \lambda_i^2 \right\}
\]

\[
= \frac{1}{\sqrt{nd}} \left\{ \left( \hat{\rho}^{-2} \sum_{i=1}^{m} \lambda_i^2 \right) \left( \sum_{i=1}^{m} \left( \lambda_i^2 + n\sigma^2 \right) \right) + m \left( n\sigma^2 - \frac{1}{p^2} \hat{\sigma}_p \right) \right\}
\]

\[
= \frac{1}{\sqrt{nd}} \left\{ (a) + m (b) \right\}.
\]

First, consider the term \( (a) \). We have

\[
(a) = \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)} T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \sum_{i=1}^{m} \left( \lambda_i^2 + n\sigma^2 \right)
\]

\[
= \left\{ \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)} T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \sum_{i=1}^{m} \left( \lambda_i^2 + n\sigma^2 \right) \right\}
\]
\[
\begin{align*}
&+ \left\{ \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}T \hat{\Sigma}_p \hat{V}^{(m)} \right) \right\} \\
&+ \left\{ \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}T \hat{\Sigma}_p \hat{V}^{(m)} \right) \right\} \\
&= (i) + (ii) + (iii).
\end{align*}
\]

By (9), there is \( O \in \mathbb{V}_{m,m} \) such that
\[
\| \hat{V}^{(m)} - V^{(m)}_p O \|_F^2 \leq 2 \| \sin(\hat{V}^{(m)}, V^{(m)}_p) \|_F^2 \quad \text{and} \quad O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p V^{(m)}_p O = \lambda^2_{pi},
\]
where \( O_i \) is the \( i \)-th column of \( O \). Then, the term \( (i) \) is
\[
(i) = \frac{1}{p^2} \text{tr} \left( O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p V^{(m)}_p O \right) - \sum_{i=1}^{m} \left[ \lambda_i^2 + n\sigma^2 \right] \\
+ \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}_p T \hat{\Sigma}_p \hat{V}^{(m)} - O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p V^{(m)}_p O \right) \\
= \frac{1}{p^2} \text{tr} \left( \hat{V}^{(m)}_p T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \sum_{i=1}^{m} \left[ \lambda_i^2 + n\sigma^2 \right] \\
+ \frac{1}{p^2} \sum_{i=1}^{m} \left( \hat{V}^{(m)}_p \hat{\Sigma}_p \hat{V}^{(m)}_p - O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p V^{(m)}_p O_i \right)
\]
\[
(21) \quad = \frac{1}{p^2} \sum_{i=1}^{m} \lambda_i^2 - \sum_{i=1}^{m} \left[ \lambda_i^2 + n\sigma^2 \right] + O_p \left( \frac{1}{pd^2} \right),
\]
where the last equality holds by the fact that
\[
\begin{align*}
\sum_{i=1}^{m} \left( \hat{V}^{(m)}_p \hat{\Sigma}_p \hat{V}^{(m)}_p - O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p V^{(m)}_p O_i \right) \\
= \sum_{i=1}^{m} \left[ (\hat{V}^{(m)}_p - V^{(m)}_p O_i) T \hat{\Sigma}_p (\hat{V}^{(m)}_p - V^{(m)}_p O_i) + 2\lambda^2_{pi} O^T \hat{V}^{(m)}_p T \hat{\Sigma}_p \hat{V}^{(m)}_p - 2\lambda^2_{pi} \right] \\
= \sum_{i=1}^{m} \left[ (\hat{V}^{(m)}_p - V^{(m)}_p O_i) T \hat{\Sigma}_p (\hat{V}^{(m)}_p - V^{(m)}_p O_i) - \lambda^2_{pi} \| \hat{V}^{(m)}_p - V^{(m)}_p O_i \|_F^2 \right] \\
\leq 2\lambda^2_{p1} \sum_{i=1}^{m} \| \hat{V}^{(m)}_p - V^{(m)}_p O_i \|_F^2 \\
= 2\lambda^2_{p1} \left\| \hat{V}^{(m)}_p - V^{(m)}_p O \right\|_F^2 \quad \text{and} \quad O_p \left( \frac{1}{pd^2} \right),
\end{align*}
\]
\[
(22) \quad = O_p \left( \frac{1}{pd^2} \right),
\]
where the last equality is due to (9), (19), and (23) below; by the application of Weyl’s theorem (Li [1998a]) and Lemma 6.1, we can show
\[
(23) \quad \lambda^2_{p1} = O_p(p^2 nd).
\]
The term \((ii)\) is

\[
\mathbb{E} \left| (\hat{\hat{p}} - p) \right| \left| \text{tr} \left( \hat{\hat{V}}^{(m)T} \text{diag}(\hat{\Sigma}) \hat{\hat{V}}^{(m)} \right) \right| \\
\leq \frac{m}{p^2} \mathbb{E} \left| (\hat{\hat{p}} - p) \right| \max_{1 \leq i \leq m} \hat{\hat{V}}^T_i \text{diag}(\hat{\hat{\Sigma}}) \hat{\hat{V}}_i \\
\leq \frac{m}{p^2} \left\{ \mathbb{E} (\hat{\hat{p}} - p)^2 \right\}^{1/2} \left\{ \mathbb{E} \left[ \max_{1 \leq i \leq m} \hat{\hat{V}}^T_i \text{diag}(\hat{\hat{\Sigma}}) \hat{\hat{V}}_i \right]^2 \right\}^{1/2} \\
\leq \frac{m}{p^2} \left\{ \mathbb{E} (\hat{\hat{p}} - p)^2 \right\}^{1/2} \left\{ \mathbb{E} \left[ \left| \text{diag}(\hat{\hat{\Sigma}}) \right|^2 \right] \right\}^{1/2} \\
= \frac{m}{p^2} \sqrt{\frac{p(1 - p)}{nd}} \left\{ \mathbb{E} \left[ \left| \text{diag}(\hat{\hat{\Sigma}}) \right|^2 \right] \right\}^{1/2} \\
(24) = O \left( \max \left\{ \frac{1}{p}, \sqrt{\frac{n}{pd}} \right\} \right),
\]

where the second inequality is due to Hölder’s inequality and the last equality holds by the fact that

\[
\mathbb{E} \left[ \left| \text{diag}(\hat{\hat{\Sigma}}) \right|^2 \right] \\
\leq 4 \mathbb{E} \left[ \left| \text{diag}(\hat{\hat{\Sigma}}) - p \text{diag}(\hat{\Sigma}) + np\sigma^2 I_d \right|^2 \right] \\
+ \left\{ \mathbb{E} \left[ \left| \text{diag}(\hat{\hat{\Sigma}}) \right|^2 \right] \right\} \\
= 4 \mathbb{E} \left[ \max_{1 \leq h \leq d} \left\{ \sum_{k=1}^n (M_{kh}^2 - pM_{0kh}^2 + p\sigma^2) \right\}^2 \right] \\
+ 4 \left\{ \max_{1 \leq h \leq d} \left\{ \sum_{k=1}^n M_{0kh}^2 + np\sigma^2 \right\} \right\}^2 \\
\leq 4 \sum_{h=1}^d \mathbb{E} \left[ \left\{ \sum_{k=1}^n (M_{kh}^2 - p(M_{0kh}^2 + \sigma^2)) \right\}^2 \right] + 4 \left\{ np(L^2 + \sigma^2) \right\}^2 \\
= 4 \sum_{h=1}^d \sum_{k=1}^n \mathbb{E} \left[ M_{kh}^2 - p(M_{0kh}^2 + \sigma^2) \right]^2 \\
+ 4 \left\{ np(L^2 + \sigma^2) \right\}^2 \\
= O \left( \max \{ pnd, p^2n^2 \} \right).
\]

The term \((iii)\) in (20) is

\[
(iii) = \left( \frac{1}{p^2} - \frac{1}{p^3} \right) \left[ \text{tr} \left( \hat{\hat{V}}^{(m)T} \hat{\hat{\Sigma}} \hat{\hat{V}}^{(m)} \right) - p^2 \sum_{i=1}^m (\lambda_i^2 + n\sigma^2) \right] \\
+ \left( \frac{1}{p^2} - \frac{1}{p^3} \right) p^2 \sum_{i=1}^m (\lambda_i^2 + n\sigma^2)
\]
\begin{align*}
\text{asy}
\frac{1}{\hat{p}^2} - \frac{1}{p^2} \left[ \text{tr} \left( \hat{V}^{(m)}^T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \text{tr} \left( \hat{V}^{(m)}^T \hat{\Sigma}_p \hat{V}^{(m)} \right) \\
+ \text{tr} \left( \hat{V}^{(m)}^T \hat{\Sigma}_p \hat{V}^{(m)} \right) - \text{tr} \left( \hat{O}^T \hat{V}^{(m)}_p^T \hat{\Sigma}_p \hat{V}^{(m)}_p \hat{O} \right) \\
+ \text{tr} \left( \hat{O}^T \hat{V}^{(m)}_p^T \hat{\Sigma}_p \hat{V}^{(m)}_p \hat{O} \right) - p^2 \sum_{i=1}^{m} \left( \lambda_i^2 + n \sigma^2 \right) \right] \\
+ \left( \frac{1}{\hat{p}_2} - \frac{1}{p_2} \right) p^2 \sum_{i=1}^{m} \left( \lambda_i^2 + n \sigma^2 \right) \\
= O_p \left( \frac{1}{\sqrt{p^3 nd}} \right) \left[ O_p \left( \max \left\{ p, \sqrt{\frac{p^3 n}{d}} \right\} \right) + O_p \left( \frac{p}{d^2} \right) + O_p \left( \sqrt{p^3 nd} \right) \right] \\
+ \left( \frac{1}{\hat{p}^2} - \frac{1}{p^2} \right) p^2 \sum_{i=1}^{m} \left( \lambda_i^2 + n \sigma^2 \right)
\end{align*}

\begin{align*}
(25) \quad & O_p \left( \frac{1}{p} \right) + \left( \frac{1}{\hat{p}^2} - \frac{1}{p^2} \right) p^2 \sum_{i=1}^{m} \left( \lambda_i^2 + n \sigma^2 \right),
\end{align*}

where the third equality is due to (24), (22), Proposition 6.2, and the fact that

\begin{align*}
(26) \quad \sqrt{nd} \left( \frac{1}{\hat{p}^2} - \frac{1}{p^2} \right) \to \mathcal{N} \left( 0, \frac{4(1-p)}{p^5} \right) \text{ in distribution, as } n, d \to \infty,
\end{align*}

by CLT and Delta method. From (21), (24), and (25), we have

\begin{align*}
(a) &= \frac{1}{p^2} \sum_{i=1}^{m} \lambda_{p_i}^2 - \sum_{i=1}^{m} \left[ \lambda_i^2 + n \sigma^2 \right] \\
&\quad + \left( \frac{1}{\hat{p}^2} - \frac{1}{p^2} \right) p^2 \sum_{i=1}^{m} \left( \lambda_i^2 + n \sigma^2 \right) + o_p \left( \sqrt{\frac{nd}{p}} \right) .
\end{align*}

Second, the term (b) is

\begin{align*}
(b) &= n \sigma^2 - \frac{1}{\hat{p}_2} \hat{\tau}_p \\
&= \left( n \sigma^2 - \frac{1}{p^2} \hat{\tau}_p \right) + \left( \frac{1}{\hat{p}_2} - \frac{1}{p^2} \right) \hat{\tau}_p + \frac{1}{p^2} (\hat{\tau}_p - \hat{\tau}_p) \\
&= O_p \left( \frac{\sqrt{n}}{p} \right) + O_p \left( \sqrt{\frac{nd}{p d}} \right) + \frac{1}{p^2} (\hat{\tau}_p - \hat{\tau}_p) \\
&= o_p \left( \sqrt{\frac{nd}{p}} \right),
\end{align*}

\begin{align*}
(28)
\end{align*}
where the third equality is due to Proposition 6.3 and (26), and the last equality holds by the fact that there is $\tilde{O} \in V_{d-r,d-r}$ by (9) such that
\[ \| \hat{V}_{c}^{(m)} - V_{pc}^{(m)} \tilde{O} \|_{F}^{2} \leq 2 \| \sin(\hat{V}_{c}^{(m)}, V_{pc}^{(m)}) \|_{F}^{2} \] and $\tilde{O}^{T} V_{pc}^{T} \tilde{O} V_{pc} \tilde{O}^{i} = \lambda_{p^{r+i}}^{2}$, where $\tilde{O}^{i}$ is the $i$-th column of $\tilde{O}$, and that
\[
\frac{1}{(d-r)} \left| \hat{\tau}_{p} - \tilde{\tau}_{p} \right| = \frac{1}{(d-r)} \left| \text{tr} \left( \hat{O}^{T} V_{pc}^{T} \hat{\Sigma}_{p} V_{pc} \hat{O} \right) - \text{tr} \left( \hat{V}_{c}^{T} \hat{\Sigma}_{p} V_{pc} \hat{O} \right) \right|
\]
\[
\leq \frac{1}{(d-r)} \left| \text{tr} \left( \hat{O}^{T} V_{pc}^{T} \hat{\Sigma}_{p} V_{pc} \hat{O} \right) - \text{tr} \left( \hat{V}_{c}^{T} \hat{\Sigma}_{p} V_{pc} \hat{O} \right) \right| + \frac{1}{(d-r)} \left| \left( \hat{\tau}_{p} - \tilde{\tau}_{p} \right) \text{tr} \left( \hat{V}_{c}^{T} \text{diag}(\hat{\Sigma}) V_{c} \right) \right|
\]
\[
= \frac{1}{(d-r)} 4 \lambda_{p1}^{2} \left( \| \sin(V_{pc}, V_{c}) \|_{F}^{2} + \frac{1}{(d-r)} \left| \left( \hat{\tau}_{p} - \tilde{\tau}_{p} \right) \text{tr} \left( \hat{V}_{c}^{T} \text{diag}(\hat{\Sigma}) V_{c} \right) \right| \right)
\]
\[
= O_{p} \left( \frac{p}{d^{3/2}} \right) + O_{p} \left( \max \left\{ p, \frac{p^{3/2}}{\sqrt{n/d}} \right\} \right),
\]
where the second inequality can be derived similarly to (22), the second equality holds similarly to (24), and the last equality is due to (19) and (23).

Combining the results in (27) and (28), we have
\[
\frac{1}{\sqrt{nd}} \left\{ \sum_{i=1}^{m} \hat{\lambda}_{i}^{2} - \sum_{i=1}^{m} \lambda_{i}^{2} \right\}
\]
\[
= \frac{1}{\sqrt{nd}} \left\{ (a) + m (b) \right\}
\]
\[
= \frac{1}{\sqrt{nd}} \left\{ \frac{1}{p^{2}} \sum_{i=1}^{m} \hat{\lambda}_{pi}^{2} - \sum_{i=1}^{m} \left[ \lambda_{i}^{2} + n\sigma^{2} \right] + \left( \frac{1}{p^{2}} - \frac{1}{p^{2}} \right) p^{2} \sum_{i=1}^{m} \left( \lambda_{i}^{2} + n\sigma^{2} \right) \right\}
\]
\[
+ o_{p}(1).
\]
Thus, by Proposition 6.2, Delta method and Slutsky’s theorem, we have
\[
\frac{1}{\sqrt{nd\sigma_{\lambda}}} \left\{ \sum_{i=1}^{m} \hat{\lambda}_{i}^{2} - \sum_{i=1}^{m} \lambda_{i}^{2} \right\} \rightarrow \mathcal{N}(0,1) \text{ in distribution, as } n,d \rightarrow \infty,
\]
where $\sigma_{\lambda}^{2} = (1 - 2p^{-3}) \Gamma_{nd} \left( \frac{1}{-2p^{-3}} \right)$. \qed
6.3. Proofs for Theorem 4.4. The proof of the following Proposition is in Appendix A.4.

Proposition 6.4. Under the model setup in Section 2, Assumption 1, and Assumption 2(2), we have
\[
\|\hat{M}(s_0) - M_0\|^2_F = \frac{1}{pb^4} O_p(n),
\]
where \(\hat{M}(s_0)\) are defined in (12) and (13) and \(M_0\) is defined in (1).

Proof of Theorem 4.4. For any given \(\eta > 0\), we have for a large \(n\),
\[
P\left(\min_{s \in \{-1,1\}} \|P_\Omega(M(s)) - P_\Omega(M)\|^2_F < \|P_\Omega(\hat{M}(s_0)) - P_\Omega(M)\|^2_F \right) \leq \eta/2
\]
by Assumption 2(1). Also, for any given \(\eta > 0\), we can find \(C_\eta > 0\), free of \(n, d,\) and \(p\), such that for large \(n\),
\[
P\left(\frac{pb^4}{n} \|\hat{M}(s_0) - M_0\|^2_F \geq C_\eta\right) \leq \eta/2
\]
by Proposition 6.4. Therefore, for any given \(\eta > 0\), we can find \(C_\eta > 0\) such that
\[
P\left(\frac{pb^4}{n} \|\hat{M}(s_0) - M_0\|^2_F \geq C_\eta\right)
\]
\[
= \frac{pb^4}{n} \|\hat{M}(s_0) - M_0\|^2_F \geq C_\eta, s_0 = \hat{s}
\]
\[
+ \frac{pb^4}{n} \|\hat{M}(s_0) - M_0\|^2_F \geq C_\eta, s_0 \neq \hat{s}
\]
\[
\leq \frac{pb^4}{n} \|\hat{M}(s_0) - M_0\|^2_F \geq C_\eta
\]
\[
+ \min_{s \in \{-1,1\}} \|P_\Omega(M(s)) - P_\Omega(M)\|^2_F < \|P_\Omega(\hat{M}(s_0)) - P_\Omega(M)\|^2_F
\]
\[
\leq \eta/2 + \eta/2
\]
\[
= \eta.
\]
Or, for any given \(\eta > 0\) and \(\zeta > 0\), there exists \(N_\zeta > 0\) such that for all \(n \geq N_\zeta\),
\[
P\left(\frac{pb^4}{h_n n} \|\hat{M}(s) - M_0\|^2_F > \eta\right)
\]
\[
\begin{align*}
&= \mathbb{P}\left( \frac{p b_r^2}{h_n n} \left\| \hat{M}(s_0) - M_0 \right\|_F^2 > \eta, s_0 = \hat{s} \right) \\
&+ \mathbb{P}\left( \frac{p b_r^4}{h_n n} \left\| \hat{M}(\hat{s}) - M_0 \right\|_F^2 > \eta, s_0 \neq \hat{s} \right) \\
&\leq \mathbb{P}\left( \frac{p b_r^2}{h_n n} \left\| \hat{M}(s_0) - M_0 \right\|_F^2 \geq \eta \right) \\
&+ \mathbb{P}\left( \min_{s \in \{-1, 1\}^r} \left\| \mathcal{P}_{\Omega}(\hat{M}(s)) - \mathcal{P}_{\Omega}(M) \right\|_F^2 < \left\| \mathcal{P}_{\Omega}(\hat{M}(s_0)) - \mathcal{P}_{\Omega}(M) \right\|_F^2 \right) \\
&\leq \frac{\zeta}{2} + \frac{\zeta}{2} = \zeta,
\end{align*}
\]

where the second inequality holds due to Assumption 2(1) and Proposition 6.4. \qed

References.


**Proof of Lemma 3.1.**

Let
\[
M_y = \left[ (y_{ij} - p)M_{0ij} \right]_{1 \leq i \leq n, 1 \leq j \leq d} \quad \text{and} \quad \epsilon_y = \left[ y_{ij} \epsilon_{ij} \right]_{1 \leq i \leq n, 1 \leq j \leq d},
\]
both in \( \mathbb{R}^{n \times d} \). Then,
\[
M = pM_0 + M_y + \epsilon_y \quad \text{and} \quad \hat{\Sigma} = p^2 M_T M_0 M_0^T + M_T M_y + \epsilon_T \epsilon_y + pM_0 M_T \epsilon_y + pM_T \epsilon_y + M_T \epsilon_y + \epsilon_T M_y.
\]
Equation (A.1) follows since under the model setup in Section 2,
\[
E M_y = 0, \quad E \epsilon_y = 0, \quad E(M_T M_y) = p(1-p) \text{diag}(M_0^T M_0), \quad E(\epsilon_T \epsilon_y) = np \sigma^2 I_d,
\]
\[
E(M_0^T M_y) = 0, \quad E(M_y \epsilon_y) = 0, \quad \text{and} \quad E(M_y \epsilon_y) = 0.
\]

We can similarly show the result (3).

**Proof of Proposition 6.1.**

We only show the result (14), since the other result can be shown similarly. Let
\[
E = \left\{ \frac{1}{nd} | \lambda_{p}^2 - \lambda_{p,m+1}^2 | < t \right\},
\]
where \( t = C_1 p \frac{\log n}{d} + C_2 p^{3/2} \sqrt{\log n} \). Note that \( \frac{t}{p^2} \rightarrow 0 \). By Weyl’s theorem (Li [1998a]) and Lemma 6.1, we have for large constants \( C_1, C_2 > 0 \),
\[
P(E^c) \leq P \left( \frac{1}{nd} \left\| \hat{\Sigma}_p - E \hat{\Sigma}_p \right\|_2 \geq t \right) = O(n^{-2}).
\]
Thus, for large \( n \),
\[
E \left\| \sin \left( V_p^{(m)}, V^{(m)} \right) \right\|_F^2 = E \left\{ \left\| \sin \left( V_p^{(m)}, V^{(m)} \right) \right\|_F^2 \mathbb{1}_{E^c} \right\} + E \left\{ \left\| \sin \left( V_p^{(m)}, V^{(m)} \right) \right\|_F^2 \mathbb{1}_E \right\}
\]
\[
\leq m P (E^c) + E \left\{ \frac{1}{nd} \left\| \hat{\Sigma}_p - E \hat{\Sigma}_p \right\|_2 \mathbb{1}_{E^c} \right\} + E \left\{ \frac{1}{nd} \left\| \lambda_p^2 - \lambda_{p,m+1}^2 \right\|_2 \mathbb{1}_E \right\}.
\]
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\begin{align*}
\leq m \mathbb{P}(E^c) + \mathbb{E} \left\{ \left( \frac{1}{nd} \frac{1}{p^2} \left( \tilde{\Sigma}_p - \mathbb{E} \tilde{\Sigma}_p \right) V^{(m)} \right)^2_{F} \right\} \\
\leq m \mathbb{P}(E^c) + \mathbb{E} \left\{ \left( \frac{1}{nd} \frac{1}{p^2} \left| \lambda_{pm} - \lambda_{pm+1}^{2} \right| - \frac{t^2}{p^2} \right)^2 \right\} \\
\leq m \mathbb{P}(E^c) + \mathbb{E} \left\{ \left( \frac{1}{2nd} \frac{1}{p^2} \left| \tilde{\lambda}_{pm} - \tilde{\lambda}_{pm+1}^{2} \right| \right)^2 \right\} \\
\leq cn^{-2} + \frac{Cn^{-1}}{p(b_m^2 - b_{m+1}^2)^2},
\end{align*}

(A.2)

where $\mathbb{1}_E$ is an indicator function of an event $E$, the first inequality is due to the fact that $\| \sin(\tilde{V}^{(m)}, V^{(m)}_p)\|_{F}^2 \leq m$ and Davis-Kahan sin $\theta$ theorem (Theorem 3.1 in Li [1998b]), and the last inequality holds by Lemma A.1 below.

**Lemma A.1.** Under the model setup in Section 2 and Assumption 1, we have for large $n$ and $d$,

\begin{align*}
\mathbb{E} \left\{ \left( \frac{1}{nd} \frac{1}{p^2} \left( \tilde{\Sigma}_p - \mathbb{E} \tilde{\Sigma}_p \right) V^{(m)} \right)^2_{F} \right\} \leq \frac{C_1}{pn}
\end{align*}

and

\begin{align*}
\mathbb{E} \left\{ \left( \frac{1}{nd} \frac{1}{p^2} \left( \tilde{\Sigma}_p - \mathbb{E} \tilde{\Sigma}_p \right) U^{(m)} \right)^2_{F} \right\} \leq \frac{C_2}{pd},
\end{align*}

where $\tilde{\Sigma}_p$ and $\tilde{\Sigma}_p$ are defined in (4) and $C_1$ and $C_2$ are generic constants free of $n, d$, and $p$.

**Proof of Lemma A.1.** We only show the result (A.3) because the other result holds similarly.

From (A.1), (4), Proposition 3.1, and triangle inequality, we have

\begin{align*}
\left\| \left( \tilde{\Sigma}_p - \mathbb{E} \tilde{\Sigma}_p \right) V^{(m)} \right\|_{F} \\
\leq \left\| \left[ M_0^T M_y - (1 - p) \text{diag}(M_0^T M_y) - p^2(1 - p) \text{diag}(M_0^T M_0) \right] V^{(m)} \right\|_{F} \\
+ \left\| \left[ \epsilon_y^T \epsilon_y - (1 - p) \text{diag}(\epsilon_y^T \epsilon_y) - np^2 \sigma^2 I_d \right] V^{(m)} \right\|_{F} \\
+ p \left\| \left[ M_0^T M_0 - (1 - p) \text{diag}(M_0^T M_0) \right] V^{(m)} \right\|_{F} \\
+ p \left\| \left[ M_0^T M_y - (1 - p) \text{diag}(M_0^T M_y) \right] V^{(m)} \right\|_{F} \\
+ p \left\| \left[ \epsilon_y^T M_0 - (1 - p) \text{diag}(\epsilon_y^T M_0) \right] V^{(m)} \right\|_{F} \\
+ p \left\| \left[ M_0^T \epsilon_y - (1 - p) \text{diag}(M_0^T \epsilon_y) \right] V^{(m)} \right\|_{F}
\end{align*}
\[ + \left\| M_y^T \epsilon_y - (1-p) \text{diag}(M_y^T \epsilon_y) \right\|_F^2 \]

\[ + \left\| \epsilon_y^T M_y - (1-p) \text{diag}(\epsilon_y^T M_y) \right\|_F^2 \]

\[ (A.4) = (A) + (B) + p(C) + p(D) + p(E) + p(F) + (G) + (H). \]

We examine the convergence rates of the above terms, (A)-(H).

First, consider the term (A) in (A.4). Then, we have

\[
\mathbb{E} \left\| M_y^T M_y - (1-p) \text{diag}(M_y^T M_y) - p^2(1-p) \text{diag}(M_0^T M_0) \right\|_F^2 \\
= \sum_{i=1}^d \sum_{j=1}^m \mathbb{E} \left\{ \sum_{k=1}^n \sum_{h=1}^d \left[ p \left( (y_{ki} - p)^2 - p(1-p) \right) M_{0ki}^2 V_{ji}^2 1_{(h=i)} \right. \right. \]
\[
\left. \left. + (y_{ki} - p)(y_{kh} - p) M_{0ki} M_{0kh} V_{jh}^2 1_{(h \neq i)} \right) \right\}^2 \]
\[
= \sum_{i=1}^d \sum_{j=1}^m \left\{ \sum_{k=1}^n \sum_{h=1}^d \left[ p^2 \mathbb{E} \left( (y_{ki} - p)^2 - p(1-p) \right) M_{0ki}^4 V_{ji}^2 1_{(h=i)} \right. \right. \]
\[
\left. \left. + \mathbb{E} \left( (y_{ki} - p)(y_{kh} - p)^2 \right) M_{0ki}^2 M_{0kh}^2 V_{jh}^2 1_{(h \neq i)} \right) \right\} \]
\[
= \sum_{i=1}^d \sum_{j=1}^m \left\{ \sum_{k=1}^n \sum_{h=1}^d \left[ p^3(1-p)(2p-1)^2 M_{0ki}^4 V_{ji}^2 1_{(h=i)} \right. \right. \]
\[
\left. \left. + p^2(1-p)^2 M_{0ki}^2 M_{0kh}^2 V_{jh}^2 1_{(h \neq i)} \right) \right\} \]
\[
\leq p^2(1-p)L^4 \sum_{i=1}^d \sum_{j=1}^m \sum_{k=1}^n \sum_{h=1}^d V_{jh}^2 \\
= p^2(1-p)L^4 \sum_{i=1}^d \sum_{j=1}^m \sum_{k=1}^n 1 \]

(A.5) \[
\leq C p^2(1-p)nd. \]

Similarly to (A.5), we can show that the expected values of the terms (B), (D), (F), (G), and (H) squared are bounded by \( C p^2 nd \).

Second, consider the term (C) in (A.4). Then, we have

\[
\mathbb{E} \left\| M_y^T M_0 - (1-p) \text{diag}(M_y^T M_0) \right\|_F^2 \\
= \sum_{i=1}^d \sum_{j=1}^m \mathbb{E} \left\{ \sum_{k=1}^n (y_{ki} - p) \sum_{h=1}^d \left[ p M_{0ki}^2 V_{ji} 1_{(h=i)} \right. \right. \]
\[
\left. \left. + (y_{ki} - p)(y_{kh} - p) M_{0ki} M_{0kh} V_{jh} 1_{(h \neq i)} \right) \right\} \]
\[
= \sum_{i=1}^d \sum_{j=1}^m \mathbb{E} \left\{ \sum_{k=1}^n \sum_{h=1}^d \left[ p^2 (y_{ki} - p)^2 M_{0ki}^4 V_{ji}^2 1_{(h=i)} \right. \right. \]
\[
\left. \left. + p^2 (y_{ki} - p)(y_{kh} - p)^2 M_{0ki}^2 M_{0kh}^2 V_{jh}^2 1_{(h \neq i)} \right) \right\} \]
\[
\leq p^2(1-p)L^4 \sum_{i=1}^d \sum_{j=1}^m \sum_{k=1}^n \sum_{h=1}^d V_{jh}^2 \\
= p^2(1-p)L^4 \sum_{i=1}^d \sum_{j=1}^m \sum_{k=1}^n 1 \]

(A.6) \[
\leq C p^2(1-p)nd. \]
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\[ + M_{0ki}M_{0kh}V_{jh} \mathbb{1}_{(h \neq i)} \]\}

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{E}(y_{ki} - p)^2 \left\{ \sum_{h=1}^{d} M_{0ki}M_{0kh}V_{jh} \left[ 1 - (1 - p)\mathbb{1}_{(h = i)} \right] \right\}^2 \]

\[ = p(1 - p) \sum_{i=1}^{d} \sum_{j=1}^{n} \left\{ \sum_{h=1}^{d} M_{0ki}M_{0kh}V_{jh} \left[ 1 - (1 - p)\mathbb{1}_{(h = i)} \right] \right\}^2 \]

\[ \leq p(1 - p)L^4 \sum_{i=1}^{d} \sum_{j=1}^{n} \left\{ \sum_{h=1}^{d} |V_{jh}| \right\}^2 \]

(A.6) \( \leq Cp(1 - p)nd^2 \),

where the last inequality holds due to Cauchy-Schwarz inequality.

Lastly, for the term (E) in (A.4),

\[ \mathbb{E} \left\| \left[ \epsilon_y^T M_0 - (1 - p)\text{diag}(\epsilon_y^T M_0) \right] V^{(m)} \right\|_F^2 \]

\[ = \sum_{i=1}^{d} \sum_{j=1}^{m} \mathbb{E} \left\{ \sum_{k=1}^{n} y_{ki} \epsilon_{ki} \sum_{h=1}^{d} M_{0kh}V_{jh} \left[ 1 - (1 - p)\mathbb{1}_{(h = i)} \right] \right\}^2 \]

\[ = \sum_{i=1}^{d} \sum_{j=1}^{m} \sum_{k=1}^{n} \mathbb{E}(y_{ki}^2 \epsilon_{ki}) \left\{ \sum_{h=1}^{d} M_{0kh}V_{jh} \left[ 1 - (1 - p)\mathbb{1}_{(h = i)} \right] \right\}^2 \]

\[ = p\sigma^2 \sum_{i=1}^{d} \sum_{j=1}^{m} \sum_{k=1}^{n} \left( \sum_{h=1}^{d} M_{0kh}V_{jh} \left[ 1 - (1 - p)\mathbb{1}_{(h = i)} \right] \right) \]

\[ \leq p\sigma^2 L^2 \sum_{i=1}^{d} \sum_{j=1}^{m} \sum_{k=1}^{n} \left( \sum_{h=1}^{d} |V_{jh}| \right) \]

(A.7) \( \leq Cpnd^2 \),

where last inequality holds due to Cauchy-Schwarz inequality.

The result follows from (A.5)-(A.7).

\[ \square \]

**Lemma A.2.** Under the model setup in Section 2 and Assumption 1, we have for any given \( \xi_1 > 0 \),

\[ \|M_y\|_2 \leq C_{\xi_1} \sqrt{pn \log n} \]

with probability \( 1 - O(n^{-\xi_1}) \). Similarly, we have for any given \( \xi_2 > 0 \),

\[ \|\epsilon_y\|_2 \leq C_{\xi_2} \sqrt{pn \log n} \]

with probability \( 1 - O(n^{-\xi_2}) \).
**Proof of Lemma A.2.** Let $M(y_{ij}) \in \mathbb{R}^{n \times d}$ be such that

$$
M(y_{kj}) = \begin{cases} 
(y_{kh} - p)M_{0kh}, & (k, h) = (i, j) \\
0, & (k, h) \neq (i, j)
\end{cases} \quad \text{for } 1 \leq k \leq n \text{ and } 1 \leq h \leq d.
$$

Then,

$$
\frac{1}{nd}M_y = \frac{1}{nd} \sum_{i=1}^{n} \sum_{j=1}^{d} M_y^{(i,j)},
$$

$E(M_{y(i,j)}) = 0$, and $\|M_{y(i,j)}\|_2 \leq L$ for all $1 \leq k \leq n$ and $1 \leq h \leq d$. Also, we have

$$
\left\| \frac{1}{nd}E\left(M_y^{(i,j)}M_y^{(i,j)T}\right) \right\|_2 = \left\| \frac{p(1-p)}{nd}\text{diag}(M_0M_0^T) \right\|_2 \leq \frac{pL^2}{n} \quad \text{and}
$$

(A.8) $$
\left\| \frac{1}{nd}E\left(M_y^{(i,j)T}M_y^{(i,j)}\right) \right\|_2 = \left\| \frac{p(1-p)}{nd}\text{diag}(M_0^TM_0) \right\|_2 \leq \frac{pL^2}{d}.
$$

Thus, by Proposition 1 in Koltchinskii et al. [2011a], we have

$$
\left\| \frac{1}{nd}M_y \right\|_2 \leq C \max\left( \sqrt{\frac{pL^2}{d}}, \sqrt{\frac{\log n}{nd}}, \sqrt{\frac{L \log n}{nd}} \right) \leq C\sqrt{\frac{p \log n \log(1/d)}{nd^2}}
$$

with probability at least $1 - n^{-\xi_1}$.

In a similar way together with Proposition 2 in Koltchinskii et al. [2011a], we can show that $\left\| \frac{1}{nd}\epsilon_y \right\|_2 \leq C\sqrt{\frac{p \log n \log(1/d)}{nd^2}}$ with probability at least $1 - n^{-\xi_2}$.

**Proof of Lemma 6.1.** We only show the result (15) because the other result holds similarly.

From (A.1), Proposition 3.1 and triangle inequality, we have

$$
\frac{1}{nd} \left\| \tilde{\Sigma}_y - \tilde{\Sigma}_y \right\|_2 \leq \frac{1}{nd} \left\| M_y^TM_y - (1-p)\text{diag}(M_y^TM_y) - p^2(1-p)\text{diag}(M_0^TM_0) \right\|_2 \\
+ \frac{1}{nd} \left\| \epsilon_y^T\epsilon_y - (1-p)\text{diag}(\epsilon_y^T\epsilon_y) - np^2\sigma^2I_d \right\|_2 \\
+ 2\frac{1}{nd} \left\| pM_y^TM_0 - (1-p)p\text{diag}(M_y^TM_0) \right\|_2 \\
+ 2\frac{1}{nd} \left\| p\epsilon_y^TM_0 - (1-p)p\text{diag}(\epsilon_y^TM_0) \right\|_2 \\
+ 2\frac{1}{nd} \left\| M_y^T\epsilon_y - (1-p)\text{diag}(M_y^T\epsilon_y) \right\|_2
$$

with probability at least $1 - n^{-\xi_1}$. 

\[\square\]
(A.9) \[ (I) + (II) + 2 (III) + 2 (IV) + 2 (V). \]

Because of similarity, we provide arguments only for (I) and (IV).

Consider the term (I) in (A.9). First, we have by Lemma A.2

\[
\frac{1}{n d} \| M_T y M_y \|_2 = n d \left\| \frac{1}{n d} M_y \right\|_2^2 \leq C p \frac{\log n}{d}
\]

with probability at least \( 1 - O(n^{-\mu_1}) \). Also, we have with probability at least \( 1 - O(n^{-\mu_1}) \),

\[
\frac{1 - p}{n d} \| \text{diag}(M_T y) + p^2 \text{diag}(M_0^T M_0) \|_2 \\
\leq \frac{1 - p}{n d} \| \text{diag}(M_T y) - p(1 - p) \text{diag}(M_0^T M_0) \|_2 \\
+ \frac{p(1 - p)}{n d} \| \text{diag}(M_0^T M_0) \|_2 \\
= (1 - p) \max_{1 \leq h \leq d} \left| \sum_{k=1}^{n} \frac{[(y_{kh} - p)^2 - p(1 - p)] M_{0kh}^2}{n d} \right| \\
+ \frac{p(1 - p)}{n d} \max_{1 \leq h \leq d} \left| \sum_{k=1}^{n} M_{0kh}^2 \right| \\
\leq C \sqrt{\frac{p \log n}{d} + \frac{p(1 - p)L^2}{d}} \\
\leq C p d^{-1},
\]

where the second inequality holds by (A.12) below. Take \( t^2 = c \frac{\log n}{nd^2} p(1 - p)(3p^2 - 3p + 1) \) for some large constant \( c > 0 \). Then, by Bernstein’s inequality,

\[
P \left( \max_{1 \leq h \leq d} \left| \sum_{k=1}^{n} \frac{[(y_{kh} - p)^2 - p(1 - p)] M_{0kh}^2}{n d} \right| \geq t \right) \\
\leq \sum_{h=1}^{d} P \left( \left| \sum_{k=1}^{n} \frac{[(y_{kh} - p)^2 - p(1 - p)] M_{0kh}^2}{n d} \right| \geq ndt \right) \\
\leq 2d \exp \left\{ - \frac{nd^2 t^2}{2L^4 p(1 - p)(3p^2 - 3p + 1)} \right\} \\
= C n^{-\mu_1}.
\]

By (A.10) and (A.11), we have

\[
(I) \leq C p \frac{\log n}{d}
\]

with probability at least \( 1 - O(n^{-\mu_1}) \). Similarly, we can show that (II) and (V) are bounded by \( C p \frac{\log n}{d} \) with probability at least \( 1 - O(n^{-\mu_1}) \).
Consider the term \((IV)\) in (A.9). We have
\[
(IV)^2 \leq \left\{ \max_{1 \leq j \leq d} \sum_{i=1}^{d} \left| \sum_{k=1}^{n} X_{kij}^{(IV)} \right| \right\} \left\{ \max_{1 \leq i \leq d} \sum_{j=1}^{d} \left| \sum_{k=1}^{n} X_{kij}^{(IV)} \right| \right\},
\]
where \(nd X_{kij}^{(IV)} = p y_{ki} c_{ki} M_{0kj} \mathbb{1}_{(i \neq j)} + p^2 y_{ki} c_{ki} M_{0kj} \mathbb{1}_{(i = j)}\) and hence \(X_{kij}^{(IV)}\) are centered sub-Gaussian random variables under the model setup in Section 2. Then, we have for any \(\rho \in \mathbb{R}\) and for all \(1 \leq k \leq n, 1 \leq i \leq d,\) and \(1 \leq j \leq d,\)
\[
E \exp \left\{ \rho X_{kij}^{(IV)} \right\} \leq \exp \left\{ \frac{\rho^2 p^3 \beta n^2 d^2}{2} \right\} \text{ for some constant } \beta > 0.
\]
Take \(t^2 = cp^3 \log n \frac{\log n}{n}\) for some large constant \(c > 0\) and \(\rho = \frac{t/d}{n p^3 \beta}.\) Then, by Markov’s inequality,
\[
\mathbb{P} \left( \max_{1 \leq j \leq d} \sum_{i=1}^{d} \left| \sum_{k=1}^{n} X_{kij}^{(IV)} \right| > t/d \right) \leq \sum_{j=1}^{d} \sum_{i=1}^{d} \mathbb{P} \left( \left| \sum_{k=1}^{n} X_{kij}^{(IV)} \right| > t/d \right) \leq 2d^2 \mathbb{E} \left\{ \exp \left\{ \rho \sum_{k=1}^{n} X_{kij}^{(IV)} \right\} \right\} \exp \left\{ \rho \left( \frac{t}{d} \right) \right\} \leq 2d^2 \exp \left\{ -\frac{t^2}{2n p^3 \beta} \right\} = C n^{-\mu_1}.
\]
(A.14)
Similarly,
\[
\mathbb{P} \left( \max_{1 \leq i \leq d} \sum_{j=1}^{d} \left| \sum_{k=1}^{n} X_{kij}^{(IV)} \right| > t \right) \leq C n^{-\mu_1}.
\]
(A.15)
By (A.14) and (A.15), with probability at least \(1 - O \left( n^{-\mu_1} \right),\)
\[
| (IV) | \leq C p^{3/2} \sqrt{\log \frac{n}{n}}.
\]
(A.16)
Similarly, we can show that \((III)\) is bounded by \(C p^{3/2} \sqrt{\log \frac{n}{n}}\) with probability at least \(1 - O(\log n^{\mu_1}).\)

The statement is showed by (A.13) and (A.16).
Proof of Lemma 6.2. We only show the result (16) because the other result holds similarly.

By triangle inequality, we have

\[
\frac{1}{nd} \left\| \hat{\Sigma} \hat{p} - \hat{\Sigma} p \right\|_2 = \frac{1}{nd} \left\| (\hat{p} - p) \text{diag}(\hat{\Sigma}) \right\|_2 \\
\leq \frac{\hat{p} - p}{nd} \left\{ \left\| \text{diag}(\hat{\Sigma}) - \text{diag}(pM_0^T M_0 + np\sigma^2 I_d) \right\|_2 \\
+ \left\| \text{diag}(pM_0^T M_0 + np\sigma^2 I_d) \right\|_2 \right\}.
\]

(A.17)

We will look at the terms in (A.17) one by one.

By Bernstein’s inequality, we have for large constant \(C > 0\),

\[
P \left( \left| \hat{p} - p \right| \geq C \sqrt{\frac{p(1-p) \log n}{nd}} \right) = P \left( \left| \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p) \right| \geq C \sqrt{p(1-p) nd \log n} \right) \\
\leq 2 \exp \left\{ -v_1 \log n \right\} = 2n^{-v_1}.
\]

(A.18)

Take \(t^2 = c^2 \frac{\log n}{np} \) for some large constant \(c > 0\). Then, since \(y_{ki}^2 (M_{0ki} + \epsilon_{ki})^2 - p(M_{0ki}^2 + \sigma^2)\), \(k = 1, \ldots, n\), are independent centered sub-exponential random variables, we have by Proposition 5.16 in Vershynin [2010],

\[
P \left( \frac{1}{nd} \left\| \text{diag}(\hat{\Sigma}) - \text{diag}(pM_0^T M_0 + np\sigma^2 I_d) \right\|_2 \geq t \right) \\
= P \left( \frac{1}{nd} \max_{1 \leq i \leq d} \left\| \sum_{k=1}^{n} \left[ y_{ki}^2 (M_{0ki} + \epsilon_{ki})^2 - p(M_{0ki}^2 + \sigma^2) \right] \right\| \geq t \right) \\
\leq \sum_{i=1}^{d} P \left( \sum_{k=1}^{n} \left| y_{ki}^2 (M_{0ki} + \epsilon_{ki})^2 - p(M_{0ki}^2 + \sigma^2) \right| \geq ndt \right) \\
\leq 2d \exp \left\{ -\frac{n^2 d^2 t^2}{c_1 np} \right\} \}
\]

(A.19) \( \leq Cn^{-v_1} \).

Also, note that

\[
\left\| \frac{1}{nd} \text{diag}(pM_0^T M_0 + np\sigma^2 I_d) \right\|_2 = \frac{1}{nd} \max_{1 \leq i \leq d} p \sum_{k=1}^{n} M_{0ki}^2 + np\sigma^2 \\
\leq \frac{p(L^2 + \sigma^2)}{d}.
\]

(A.20)
Combining the results in (A.17)-(A.20), we have

\[
\frac{1}{nd} \left\| \hat{\Sigma}_p - \hat{\Sigma}_{\hat{p}} \right\|_2 \leq Cp^{3/2} \sqrt{\frac{\log n}{nd}} \frac{1}{d}
\]

with probability at least \(1 - O(n^{-\nu_1})\).

**Proof of Lemma 6.3.** We only show the result (17) because the other result holds similarly.

We have

\[
\mathbb{E} \left\| \frac{1}{nd} \left( \hat{\Sigma}_p - \hat{\Sigma}_{\hat{p}} \right) V^{(m)} \right\|_F^2 \\
\leq m \mathbb{E} \left\| \frac{1}{nd} \left( \hat{\Sigma}_p - \hat{\Sigma}_{\hat{p}} \right) \right\|_2^2 \\
\leq m \mathbb{E} \left\{ (\hat{p} - p)^2 \left\| \frac{1}{nd} \text{diag}(\hat{\Sigma}) \right\|_2^2 \right\} \\
\leq 4m \mathbb{E} \left\{ (\hat{p} - p)^2 \left\| \frac{1}{nd} \text{diag}(\hat{\Sigma}) - \frac{1}{nd} \text{diag}(pM_0^T M_0 + n\sigma^2 I_d) \right\|_2^2 \right\} \\
+ 4m \left\| \frac{1}{nd} \text{diag}(pM_0^T M_0 + n\sigma^2 I_d) \right\|_2^2 \mathbb{E} (\hat{p} - p)^2 \\
\leq 4m \sqrt{\mathbb{E} (\hat{p} - p)^4} \mathbb{E} \left\| \frac{1}{nd} \text{diag}(\hat{\Sigma}) - \frac{1}{nd} \text{diag}(pM_0^T M_0 + n\sigma^2 I_d) \right\|_2^4 \\
+ 4m p^2 (L^2 + \sigma^2)^2 p(1 - p) \frac{d^2}{n^d} \frac{d^2}{n^d} \\
\leq C_1 p^2 (1 - p) + C_2 p^3 (1 - p) \frac{d^2}{n^d}
\]

where the fourth inequality holds by Hölder’s inequality and the fifth inequality is due to the fact that

\[
\mathbb{E} (\hat{p} - p)^4 \mathbb{E} \left\| \frac{1}{nd} \text{diag}(\hat{\Sigma}) - \frac{1}{nd} \text{diag}(pM_0^T M_0 + n\sigma^2 I_d) \right\|_2^4 \leq \mathbb{E} \left\{ \sum_{k=1}^n \sum_{h=1}^d (y_{kh} - p) \right\}^4 \\
\leq \mathbb{E} \left\{ \max_{1 \leq i \leq d} \left[ \sum_{k=1}^n \left( y_{ki}^2 (M_{0ki} + \epsilon_{ki})^2 - p(M_{0ki}^2 + \sigma^2) \right) \right] \right\}^4 \frac{d}{n^d} \frac{d}{n^d}
\leq \mathbb{E} \left\{ \sum_{k=1}^n \sum_{h=1}^d (y_{kh} - p) \right\}^4 \\
d \mathbb{E} \left[ \sum_{k=1}^n \left( y_{ki}^2 (M_{0ki} + \epsilon_{ki})^2 - p(M_{0ki}^2 + \sigma^2) \right) \right]^4 \frac{d}{n^d} \frac{d}{n^d}
\]
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\[(A.23) = \frac{O \left( p^2 (1 - p)^2 n^2 d^2 \right)}{n^8 d^8}. \]

A.3. Appendix for Section 6.2.

**Lemma A.3.** Under the model setup in Section 2 and Assumption 1, we have

\[
\sum_{i=1}^{m} \lambda_{p_i}^2 - p^2 \left[ \sum_{i=1}^{m} \lambda_i^2 + n \sigma^2 \right] = 2p \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p)M_{0kh} \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'}V_{ih'} \right) - (1 - p)M_{0kh}V_{ih} \right] + 2p \sum_{k=1}^{n} \sum_{h=1}^{d} y_{kh} \epsilon_{kh} \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'}V_{ih'} \right) - (1 - p)M_{0kh}V_{ih} \right] + O_p \left( \sqrt{nd} \right) = (i) + (ii) + O_p \left( p^2 n^2 d^2 \right),
\]

where \( \lambda_{p_i} \) and \( \lambda_i \) are defined in (4) and (1), respectively.

**Proof of Lemma A.3.** We have

\[
\sum_{i=1}^{m} \lambda_{p_i}^2 - p^2 \left[ \sum_{i=1}^{m} \lambda_i^2 + n \sigma^2 \right] = \text{tr} \left( V_p^{(m)T} \Sigma_p V_p^{(m)} \right) - \text{tr} \left( V^{(m)T} (p^2 M_0^T M_0 + np^2 \sigma^2 I_d) V^{(m)} \right) = \text{tr} \left( O^{(m)T} \Sigma_p V_p^{(m)} \right) - \text{tr} \left( V^{(m)T} (p^2 M_0^T M_0 + np^2 \sigma^2 I_d) V^{(m)} \right) + \text{tr} \left( V^{(m)T} (\epsilon^T \epsilon - (1 - p) \text{diag}(\epsilon^T \epsilon)) V^{(m)} \right) + \text{tr} \left( V^{(m)T} (p^2 M_0^T M_0 + np^2 \sigma^2 I_d) V^{(m)} \right) = \text{tr} \left( V^{(m)T} (M_y^T M_y - (1 - p) \text{diag}(M_y^T M_y) - p^2 (1 - p) \text{diag}(M_0^T M_0)) V^{(m)} \right) + \text{tr} \left( V^{(m)T} (\epsilon^T \epsilon - (1 - p) \text{diag}(\epsilon^T \epsilon)) V^{(m)} \right) + \text{tr} \left( V^{(m)T} (p^2 M_0^T M_0 + np^2 \sigma^2 I_d) V^{(m)} \right) + \text{tr} \left( V^{(m)T} p M_0^T M_y + p M_y^T M_0 \right)
\]
\begin{align*}
-(1-p) \text{diag}(M_0^T M_y + M_y^T M_0)) V^{(m)} \bigg) \\
+ \text{tr} \bigg( V^{(m)^T} (p M_0^T \epsilon_y + p \epsilon_y^T M_0 \bigg) V^{(m)} \bigg) \\
+ \text{tr} \bigg( V^{(m)^T} (M_y^T \epsilon_y + \epsilon_y^T M_y \bigg) V^{(m)} \bigg) \\
+ \text{tr}(V_p^{(m)^T} \hat{\Sigma}_p V_p^{(m)} - \mathcal{O}^T V^{(m)^T} \hat{\Sigma}_p V^{(m)} \mathcal{O}) \\
(A.24) &= (a) + (b) + (c) + (d) + (e) + (f),
\end{align*}

where $\mathcal{O} \in V_{m,m}$ is a solution to $\inf_{\mathcal{Q} \in V_{m,m}} \|V_p^{(m)} - V^{(m)} \mathcal{Q}\|_F$ and the fourth equality holds by (4) and (A.1). Below, we examine the six terms (a)-(f) one by one.

The term (a) in (A.24) is
\begin{align*}
(a) &= \sum_{i=1}^{m} V_i^T \left( M_y^T M_y - (1-p) \text{diag}(M_y^T M_y) - p^2 (1-p) \text{diag}(M_0^T M_0) \right) V_i \\
&= \sum_{i=1}^{m} \left\{ \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p) M_{0kh} V_{ih} \right\}^2 \\
&\quad - (1-p) \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p)^2 M_{0kh}^2 V_{ih}^2 \\
&\quad - p^2 (1-p) \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh}^2 V_{ih}^2 \\
&= \sum_{k=1}^{n} \sum_{h=1}^{d} p \left[ (y_{kh} - p)^2 - p(1-p) \right] M_{0kh}^2 \sum_{i=1}^{m} V_{ih}^2 \\
&\quad + 2 \sum_{k=1}^{n} \sum_{h<h'}^{1\sim d} (y_{kh} - p) (y_{kh'} - p) M_{0kh} M_{0kh'} \sum_{i=1}^{m} V_{ih} V_{ih'}. \\
(A.25)
\end{align*}

Note that the two terms in (A.25) are centered and uncorrelated with each other. So, the variance is
\begin{align*}
\text{var}(a) &= \left\{ \sum_{k=1}^{n} \sum_{h=1}^{d} p^3 (1-p)(2p-1)^2 M_{0kh}^4 \left( \sum_{i=1}^{m} V_{ih}^2 \right)^2 \right\} \\
&\quad + \left\{ 4 \sum_{k=1}^{n} \sum_{h<h'}^{1\sim d} p^2 (1-p)^2 M_{0kh}^2 M_{0kh'}^2 \left( \sum_{i=1}^{m} V_{ih} V_{ih'} \right)^2 \right\}
\end{align*}
\[
\leq m \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{h=1}^{d} p^3(1-p)(2p-1)^2 M_{0kh}^4 V_{ih}^4
\]
\[
+ 4m \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{h=1}^{1 \sim d} p^2(1-p)^2 M_{0kh}^2 M_{0kh'}^2 V_{ih}^2 V_{i'h'}^2
\]
\[
\leq mL^4 p^3(1-p)(2p-1)^2 \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{h=1}^{d} V_{ih}^4
\]
\[
+ 4mL^4 p^2(1-p)^2 \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{h,h'=1}^{1 \sim d} V_{ih}^2 V_{i'h'}^2
\]
\[
(A.26) \quad \leq Cp^2(1-p)n,
\]
where the first inequality is due to Jensen’s inequality. This shows that the term \((a)\) is \(O_p(p\sqrt{n})\). Similarly, we can show that the terms \((b)\) and \((c)\) are \(O_p(p\sqrt{n})\).

The term \((c)\) in \((A.24)\) is
\[
\frac{1}{2p}(c)
\]
\[
= \sum_{i=1}^{m} V_i^T \left( M_0^T M_y - (1-p)\text{diag}(M_0^T M_y) \right) V_i
\]
\[
= \sum_{i=1}^{m} \left\{ \sum_{k=1}^{n} \left( \sum_{h=1}^{d} M_{0kh} V_{ih} \right) \left( \sum_{h'=1}^{d} (y_{kh'} - p) M_{0kh'} V_{i'h'} \right) \right.
\]
\[
- (1-p) \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p) M_{0kh}^2 V_{ih}^2 \left\}
\]
\[
= \sum_{k=1}^{n} \sum_{h=1}^{d} (y_{kh} - p) M_{0kh} \sum_{i=1}^{m} V_i \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{i'h'} \right) - (1-p)M_{0kh} V_{ih} \right].
\]

Then, its variance is
\[
\left( \frac{1}{2p} \right)^2 \text{var}(c)
\]
\[
= \sum_{k=1}^{n} \sum_{h=1}^{d} p(1-p)M_{0kh}^2 \left\{ \sum_{i=1}^{m} V_i \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{i'h'} \right) - (1-p)M_{0kh} V_{ih} \right] \right\}^2
\]
\[
\leq Cp(1-p)nd,
\]
where the last inequality is due to Assumption 1(1) and the fact that
\[
\sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh}^2 \left\{ \sum_{i=1}^{m} V_i \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{i'h'} \right) - (1-p)M_{0kh} V_{ih} \right] \right\}^2
\]
\[ \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh}^2 \left\{ \sum_{i=1}^{m} \lambda_i U_{ik} V_{ih} - (1 - p) \sum_{i=1}^{m} M_{0kh} V_{ih}^2 \right\}^2 \]

\[ = \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh}^2 \left\{ \sum_{i=1}^{m} \lambda_i U_{ik} V_{ih} \right\}^2 \]

\[ + (1 - p)^2 \sum_{k=1}^{n} \sum_{h=1}^{d} \left\{ \sum_{i=1}^{m} M_{0kh}^2 V_{ih}^2 \right\} \]

\[ - 2(1 - p) \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh} \left\{ \sum_{i=1}^{m} \lambda_i U_{ik} V_{ih} \right\} \left\{ \sum_{i=1}^{m} M_{0kh}^2 V_{ih}^2 \right\} \]

\[ = \sum_{k=1}^{n} \sum_{h=1}^{d} M_{0kh}^2 \left\{ \sum_{i=1}^{m} \lambda_i U_{ik} V_{ih} \right\}^2 + O(n) \]

(A.27) \[ = O(nd). \]

The term (d) in (A.24) is

\[ \frac{1}{2p} (d) = \sum_{i=1}^{m} V_i^T \left( M_0^T \epsilon_y - (1 - p) \text{diag}(M_0^T \epsilon_y) \right) V_i \]

\[ = \sum_{i=1}^{m} \left\{ \sum_{k=1}^{n} \left( \sum_{h=1}^{d} M_{0kh} V_{ih} \right) \left( \sum_{h'=1}^{d} y_{kh'} \epsilon_{kh'} V_{ih'} \right) \right. \]

\[ - (1 - p) \sum_{k=1}^{n} \sum_{h=1}^{d} y_{kh} \epsilon_{kh} M_{0kh} V_{ih}^2 \}

\[ = \sum_{k=1}^{n} \sum_{h=1}^{d} y_{kh} \epsilon_{kh} \left\{ \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \right\}. \]

Then, its variance is

\[ \left( \frac{1}{2p} \right)^2 \text{var}(d) = \sum_{k=1}^{n} \sum_{h=1}^{d} p \sigma^2 \left\{ \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \right\}^2 \]

\[ \leq C p n d, \]

where the last inequality is due to Assumption 1(1) and the fact that

\[ \sum_{k=1}^{n} \sum_{h=1}^{d} \left\{ \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \right\}^2 \]

\[ = \sum_{k=1}^{n} \sum_{h=1}^{d} \left\{ \sum_{i=1}^{m} \lambda_i U_{ik} V_{ih} - (1 - p) \sum_{i=1}^{m} M_{0kh} V_{ih}^2 \right\}^2 \]
\[
\sum_{i=1}^{m} \lambda_i^2 + (1-p)^2 \sum_{k=1}^{n} \sum_{h=1}^{d} \left( \sum_{i=1}^{m} M_{0kh} V_{ih}^2 \right)^2 \\
- 2(1-p) \sum_{i=1}^{m} \lambda_i^2 \sum_{h=1}^{d} V_{ih}^2 \sum_{i'=1}^{m} V_{i'h}^2 \\
(A.28) = \sum_{i=1}^{m} \lambda_i^2 + O(n).
\]

The term \((f)\) in \((A.24)\) is
\[
|\langle f \rangle| = |\text{tr} (V_p^{(m)T} \Sigma_p V_p^{(m)} - \mathcal{O}^T V_p^{(m)} \Sigma_p V_p^{(m)} \mathcal{O})| \\
\leq \sum_{i=1}^{m} |\mathcal{O}_i^T V_p^{T} \Sigma_p V_p \mathcal{O}_i - V_p^{T} \Sigma_p V_p | \\
= \sum_{i=1}^{m} \left\{ \left| (V \mathcal{O}_i - V_{p_i})^T \Sigma_p (V \mathcal{O}_i - V_{p_i}) + 2 \lambda^2_{p_i} V_{p_i}^T (V \mathcal{O}_i - V_{p_i}) \right| \right\} \\
\leq \sum_{i=1}^{m} \lambda_{p_1}^2 \left( \| V \mathcal{O}_i - V_{p_i} \|_2^2 + 2 \| V_{p_i}^T (V \mathcal{O}_i - V_{p_i}) \| \right) \\
= \sum_{i=1}^{m} \lambda_{p_1}^2 \left( \| V \mathcal{O}_i - V_{p_i} \|_2^2 \\
+ |\mathcal{O}_i^T V_p^T V \mathcal{O}_i - \mathcal{O}_i^T V_p^T V_{p_i} - V_{p_i}^T V \mathcal{O}_i + V_{p_i}^T V_{p_i}| \right) \\
= \sum_{i=1}^{m} 2\lambda_{p_1}^2 \| V \mathcal{O}_i - V_{p_i} \|_2^2 \\
= 2\lambda_{p_1}^2 \left\| V^{(m)} \mathcal{O} - V_p^{(m)} \mathcal{O} \right\|_F^2 \\
(A.29) = O_p(pd),
\]

where \(\mathcal{O}_i\) is the \(i\)-th column of \(\mathcal{O}\) and the last equality holds by Proposition 6.1, (9), and (23).

Therefore, the result follows from \((A.24)-(A.29)\). \(\square\)

**Proof of Proposition 6.2.** By Cramèr-Wold device, it is enough to show that for any given \((c_1, c_2)^T \in \mathbb{R}^2 \setminus (0,0)^T\),
\[
\frac{1}{\sqrt{nd\gamma_{c_1,c_2}}} \left( c_1 \right)^T \left[ \left( p^2 \sum_{i=1}^{m} \lambda_{p_i}^2 \right) - \left( \sum_{i=1}^{m} [\lambda_i^2 + n\sigma^2] \right) \right] \\
\rightarrow \mathcal{N}(0,1) \text{ in distribution, as } n,d \to \infty,
\]
where \( \gamma_{c_1,c_2}^2 = (c_1 \, c_2) \Gamma_{nd} \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) \). When \( c_1 = 0 \), this can be directly showed by CLT. Thus, we only consider the case where \( c_1 \neq 0 \).

We have

\[
\left( \begin{array}{c} c_1 \\ c_2 \end{array} \right) ^T \left[ \left( \begin{array}{c} p^{-2} \sum_{i=1}^m \lambda_i^2 \\ p^2 \sum_{i=1}^m (\lambda_i^2 + n \sigma^2) \end{array} \right) - \left( \begin{array}{c} \sum_{i=1}^m [\lambda_i^2 + n \sigma^2] \\ p^3 \sum_{i=1}^m (\lambda_i^2 + n \sigma^2) \end{array} \right) \right] = c_1 \frac{1}{p^2} \sum_{i=1}^m \left[ \lambda_i^2 - p^2 (\lambda_i^2 + n \sigma^2) \right] + c_2 p^2 \sum_{i=1}^m (\lambda_i^2 + n \sigma^2) (\hat{p} - p) \\
= \frac{2c_1}{p} \sum_{k=1}^n \sum_{h=1}^d (y_{kh} - p) M_{0kh} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h' = 1}^d M_{0khh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \\
+ \frac{2c_1}{p} \sum_{k=1}^n \sum_{h=1}^d y_{kh} \epsilon_{kh} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h' = 1}^d M_{0khh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \\
+ o_p \left( \sqrt{\frac{nd}{p}} \right) + \frac{c_2 p^2}{nd} \sum_{k=1}^n \sum_{h=1}^d (y_{kh} - p) \sum_{i=1}^m (\lambda_i^2 + n \sigma^2) \\
= \sum_{k=1}^n \sum_{h=1}^d (y_{kh} - p) \left\{ \frac{2c_1}{p} M_{0kh} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h' = 1}^d M_{0khh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \\
+ \frac{2c_1}{p} M_{0kh} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h' = 1}^d M_{0khh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \right\} \\
+ o_p \left( \sqrt{\frac{nd}{p}} \right) \\
(A.30) = (a) + (b) + o_p \left( \sqrt{\frac{nd}{p}} \right) ,
\]

where the second equality holds by Lemma A.3. Since the terms \((a)\) and \((b)\) are centered and not correlated with each other under the model setup in Section 2, we have

\[
\text{var} [(a) + (b)] = \text{var} [(a)] + \text{var} [(b)] \\
= \sum_{k=1}^n \sum_{h=1}^d \mathbb{E}(y_{kh} - p)^2 \left\{ \frac{2c_1}{p} M_{0kh} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h' = 1}^d M_{0khh'} V_{ih'} \right) \right] \right\}
\]
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\[-(1 - p) M_{0kh} V_{ih} + \frac{c_2 p^2}{nd} \sum_{i=1}^{m} (\lambda_i^2 + n\sigma^2) \}\]
\[= \sum_{k=1}^{n} \sum_{h=1}^{d} \mathbb{E} \left( y_{kh} - p \right) \left\{ \frac{2c_1}{p} M_{0kh} \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] + \frac{c_2 p^2}{nd} \sum_{i=1}^{m} (\lambda_i^2 + n\sigma^2) \right\}
\[+ y_{kh} \epsilon_{kh} \left\{ \frac{2c_1}{p} \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) - (1 - p) M_{0kh} V_{ih} \right] \right\}^3 \]
\[\leq 8 \sum_{k=1}^{n} \sum_{h=1}^{d} \mathbb{E} \left| y_{kh} - p \right|^3 \frac{2c_1}{p} M_{0kh} \sum_{i=1}^{m} V_{ih} \left[ \left( \sum_{h'=1}^{d} M_{0kh'} V_{ih'} \right) \right] \]

where the third equality is due to (A.27), (A.28) and Assumption 1(1). Note that

\[(A.32) \quad nd \left( c_1 c_2 \right) \Gamma_{nd} \left( \frac{c_1}{c_2} \right) \geq \frac{4c_1^2 \sigma^2}{p} \sum_{i=1}^{m} \lambda_i^2 \geq \frac{c nd}{p} .

Thus, Liapunov’s condition is satisfied with \((a) + (b)\) because we have
\[ -(1 - p)M_{0kh}V_{ih} + O(1) \]
\[ + \mathbb{E}|y_{kh}e_{kh}|^3 \left| \frac{2c_1}{p} \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h'=1}^d M_{0kh'}V_{ih'} \right)^3 - (1 - p)M_{0kh}V_{ih} \right] \right|^3 \]
\[ \leq \frac{C}{p^2} \sum_{k=1}^n \sum_{h=1}^d \left\{ \sum_{i=1}^m V_{ih} \left[ \left( \sum_{h'=1}^d M_{0kh'}V_{ih'} \right)^3 - (1 - p)M_{0kh}V_{ih} \right] \right\}^3 \]
\[ \leq \frac{C}{p^2} \sum_{k=1}^n \sum_{h=1}^d \left\{ \sum_{i=1}^m |V_{ih}|^3 \left[ \left( \sum_{h'=1}^d M_{0kh'}V_{ih'} \right)^3 + |V_{ih}|^3 \right] + O(1) \right\} \]
\[ \leq \frac{C}{p^2} \sum_{k=1}^n \sum_{h=1}^d \left\{ \sum_{i=1}^m \sum_{h'=1}^d M_{0kh'}V_{ih'}^3 + O(nd) \right\} \]
\[ (A.33) = O \left( \frac{nd^{3/2}}{p^2} \right), \]

where the first inequality holds by Assumption 1(1), and the last two lines are due to Cauchy-Schwarz inequality.

By (A.30)-(A.33), Liapunov CLT and Slutsky theorem, we have
\[ \frac{1}{\sqrt{ndc_1 c_2}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \left( \left[ \frac{p^2}{\sum_{i=1}^m \lambda^2_{pi}} + \frac{n\sigma^2}{\sum_{i=1}^m (\lambda^2_{pi} + n\sigma^2)} \right] - \left[ \frac{1}{p^3 \sum_{i=1}^m (\lambda^2_{pi} + n\sigma^2)} \right] \right) \]
\[ \to \mathcal{N}(0, 1) \text{ in distribution, as } n, d \to \infty. \]

Proof of Proposition 6.3. Similarly to the proof of (A.24), we have
\[
\hat{\tau}_p - np^2 \sigma^2 = \frac{1}{d - r} \text{tr} \left( V_{pc}^T \hat{\Sigma}_p V_{pc} \right) - \frac{1}{d - r} \text{tr} \left( V_{pc}^T \Sigma_p V_{pc} \right)
\]
\[
= \frac{1}{d - r} \text{tr} \left( V_{pc}^T \hat{\Sigma}_p V_{pc} \right) + \frac{1}{d - r} \text{tr} \left( V_{pc}^T \Sigma_p V_{pc} - \mathcal{O}^T V_{pc} \hat{\Sigma}_p V_{pc} \mathcal{O}^T \right) - \frac{1}{d - r} \text{tr} \left( V_{pc}^T \Sigma_p V_{pc} \right)
\]
\[
= \frac{1}{d - r} \text{tr} \left( V_{pc}^T (M_y^T M_y - (1 - p) \text{diag}(M_y^T M_y)) \right) + \text{tr} \left( \mathcal{O}^T V_{pc} \hat{\Sigma}_p V_{pc} \mathcal{O}^T \right)
\]
\[
= \frac{1}{d - r} \text{tr} \left( V_{pc}^T (M_y^T M_y - (1 - p) \text{diag}(M_y^T M_y)) \right) + \text{tr} \left( \mathcal{O}^T V_{pc} \Sigma_p V_{pc} \mathcal{O}^T \right).
\]
\[ -p^2(1 - p) \text{diag}(M_0^T M_0) V_c \]
\[ + \frac{1}{d - r} \text{tr} \left( V_c^T (\epsilon_y^T \epsilon_y - (1 - p) \text{diag}(\epsilon_y^T \epsilon_y)) V_c - np^2 \sigma^2 I_{d-r} \right) \]
\[ -2p(1 - p) \frac{1}{d - r} \text{tr} \left( V_c^T (\text{diag}(M_y^T M_0)) V_c \right) \]
\[ -2p(1 - p) \frac{1}{d - r} \text{tr} \left( V_c^T (\text{diag}(\epsilon_y^T M_0)) V_c \right) \]
\[ + 2 \frac{1}{d - r} \text{tr} \left( V_c^T (M_y^T \epsilon_y - (1 - p) \text{diag}(M_y^T \epsilon_y)) V_c \right) \]
\[ + \frac{1}{d - r} \text{tr} \left( V_{pc}^T \hat{\Sigma}_p V_{pc} - O^T V_c^T \hat{\Sigma}_p V_c O \right) \]
\[ = (A) + (B) - 2(1 - p) \cdot (C) - 2(1 - p) \cdot (D) + 2 \cdot (E) + (F), \]

where \( O \in \mathbb{V}_{d-r,d-r} \) is a solution to \( \inf_{Q \in \mathbb{V}_{d-r,d-r}} \| V_{pc} - V_c Q \|_F^2 \), and the third equality is due to the fact that \( M_0 V_c = U \Lambda V^TV_c = 0 \). We will show that (A)-(F) are \( O_p \left( p \sqrt{n} \right) \).

Since the first five terms, (A)-(E), are centered, we only need to check their variances to find their rates. The variances of the terms (A), (B), and (E) are \( O \left( p^2 n \right) \), which can be shown similarly to the proof of (A.26). The variance of the term (C) is

\[
\text{var}(C) \leq \frac{1}{d - r} \sum_{i=1}^{d-r} \mathbb{E} \left[ \left( V_{ci}^T (\text{diag}(M_y^T M_0)) V_{ci} \right)^2 \right] \\
= \frac{1}{d - r} \sum_{i=1}^{d-r} \text{var} \left[ \sum_{k=1}^{d-r} \sum_{h=1}^{d-r} (y_{kh} - p) M_{0kh}^2 V_{cih}^2 \right] \\
= \frac{1}{d - r} \sum_{i=1}^{d-r} \left[ L^4 \sum_{k=1}^{n} O(p(1 - p)) \right] \\
= O(p n),
\]

where the inequality is due to Jensen's inequality. Similarly, the variance of the term (D) is \( O(p n) \).

Now, consider the term (F). Similarly to the proof of (A.29),

\[
| (F) | \leq \frac{1}{d - r} \left[ \text{tr} \left( V_{pc}^T \hat{\Sigma}_p V_{pc} - O^T V_c^T \hat{\Sigma}_p V_c O \right) \right] \\
\leq \frac{1}{d - r} \sum_{i=1}^{d-r} \left| V_{pc}^T \hat{\Sigma}_p V_{pc} - O^T V_c^T \hat{\Sigma}_p V_c O \right| \\
\leq \frac{1}{d - r} \cdot 2 \lambda_{p1}^2 \| V_{pc} - V_c O \|^2_F \\
\leq \frac{1}{d - r} \cdot 4 \lambda_{p1}^2 \| \sin (V_{pc}, V_c) \|^2_F
\]
where $O_i$ is the $i$-th column of $O$, the third inequality can be derived similarly to the proof of (22), and the last equality holds by Proposition 6.1 and (23).

\section*{A.4. Appendix for Section 6.3.}

\textbf{Proof of Proposition 6.4.} Let $\Delta \lambda_i = \hat{\lambda}_i - \lambda_i$, $\Delta U_i = \text{sign}(\langle \hat{U}_i, U_i \rangle) \hat{U}_i - U_i$, and $\Delta V_i = \text{sign}(\langle \hat{V}_i, V_i \rangle) \hat{V}_i - V_i$ for all $i \in \{1, \ldots, r\}$. Similarly to the proof of Theorem 4.2, we can show that for all $i = 1, \ldots, r$, 

(A.34) \quad |\Delta \lambda_i| = O_p \left( \frac{1}{\sqrt{p}} + \frac{1}{p} \sqrt{\frac{d}{n}} \right).

Then,

\[
\left\| \hat{M}(s_0) - M_0 \right\|_F^2 \\
= \left\| \sum_{i=1}^{r} s_{0i} \hat{\lambda}_i \hat{U}_i \hat{V}_i^T - \sum_{i=1}^{r} \lambda_i U_i V_i^T \right\|_F^2 \\
\leq r^2 \sum_{i=1}^{r} \left\| s_{0i} \hat{\lambda}_i \hat{U}_i \hat{V}_i^T - \lambda_i U_i V_i^T \right\|_F^2 \\
= r^2 \sum_{i=1}^{r} \left\| (\lambda_i + \Delta \lambda_i) \langle U_i + \Delta U_i \rangle (V_i + \Delta V_i)^T - \lambda_i U_i V_i^T \right\|_F^2 \\
\leq C r^2 \sum_{i=1}^{r} \left\{ \left\| \Delta \lambda_i U_i V_i^T \right\|_F^2 + \left\| \lambda_i \Delta U_i V_i^T \right\|_F^2 + \left\| \lambda_i U_i \Delta V_i^T \right\|_F^2 \right\} \\
= C r^2 \sum_{i=1}^{r} \left\{ O_p \left( \frac{1}{\sqrt{p}} + \frac{1}{p} \sqrt{\frac{d}{n}} \right) + O \left( nd \right) \frac{1}{p} b_r^4 O_p \left( \frac{1}{d} \right) + O \left( nd \right) \frac{1}{p} b_r^4 O_p \left( \frac{1}{n} \right) \right\} \\
= \frac{1}{p} b_r^4 O_p(n),
\]

where the third equality holds due to (A.34) and Theorem 4.1. \hfill \qed