Problem 1: Maximize the area of a rectangle inside any right triangle with base=B and height=H.

Part 1.a: Find the dimensions w and h of the rectangle and the area of the rectangle with maximum area that can fit inside the right triangle

Area of the rectangle

\[ \text{areaRectangle} := w \cdot h \]  
\[ \text{areaRectangle} := w \cdot h \]  
(1)

Equation of the line for the hypotenuse of the triangle

\[ \text{line} := y = - \frac{H}{B} \cdot x + H \]  
\[ \text{line} := y = - \frac{H}{B} \cdot x + H \]  
(2)

Define \( h \) in terms of \( w \)

\[ h_{\text{InTermsOfW}} := \text{subs}(x = w, y = h, \text{line}) \]  
\[ h_{\text{InTermsOfW}} := \frac{H w}{B} + H \]  
(3)

Sub back into area of rectangle so it now only has one variable, \( w \)

\[ \text{areaRectangleW} := \text{subs}(h_{\text{InTermsOfW}}, \text{areaRectangle}) \]  
\[ \text{areaRectangleW} := w \left( - \frac{H w}{B} + H \right) \]  
(4)

To find the maximum area, differentiate the area of the rectangle, set the derivative equal to 0, and solve for \( w \)

\[ \text{deriv} := \text{diff}(\text{areaRectangleW}, w) \]  
\[ \text{deriv} := - \frac{2 H w}{B} + H \]  
(5)

The value of \( w \) for the area of the rectangle with maximum area:

\[ w_{\text{Max}} := \text{solve}(\text{deriv} = 0, w) \]  
\[ w_{\text{Max}} := \frac{1}{2} \cdot B \]  
(6)

The value of \( h \) for the area of the rectangle with maximum area:

\[ h_{\text{Max}} := \text{subs}(w = w_{\text{Max}}, \text{rhs}(h_{\text{InTermsOfW}})) \]  
\[ h_{\text{Max}} := \frac{1}{2} \cdot H \]  
(7)

The area of the rectangle with maximum area:

\[ \text{areaRectangleMax} := \text{subs}(w = w_{\text{Max}}, \text{areaRectangleW}) \]  
\[ \text{areaRectangleMax} := \frac{1}{4} \cdot B \cdot H \]  
(8)

Part 1.b: How does the area of the rectangle compare to the area of the triangle?

The area of the triangle:

\[ \text{areaTriangle} := \frac{1}{2} \cdot B \cdot H \]
areaTriangle := \( \frac{1}{2} BH \) \hspace{1cm} (9)

The rectangle of maximum area is exactly half the area of the triangle.

\[
\text{areaRatio} := \frac{\text{areaRectangleMax}}{\text{areaTriangle}}
\]

\[
\text{areaRatio} := \frac{1}{2}
\] \hspace{1cm} (10)

Part 1.c: Solve the problem graphically.

\[
\text{areaR} := \text{subs}(H = 20, B = 15, \text{areaRectangleW})
\]

\[
\text{areaR} := w \left( -\frac{4}{3} w + 20 \right)
\] \hspace{1cm} (11)

From the plot below, we can see that the maximum area is achieved when \( w \) is half the value of \( B \).

> plot(areaR, w = 0 .. 15)

Problem 2: Volume of revolution problem

> restart

Expression for the shape of the mixer:
> \[ rr := \frac{7z(z+1)}{1+4z^2} \]

Visualize the mixer by plotting it in 3D, using parametric equations
> \[ plot3d([z, rr \cdot \cos(t), rr \cdot \sin(t)], z = 0..10, t = 0..2\cdot\Pi, scaling = constrained) \]
> with(plots):
> animate(plot3d, [[z, rr*cos(t), rr*sin(t)], z = 0 .. 10, t = 0 .. T, scaling = constrained], T = 0 .. 2*Pi)
Part 2.a: What is the volume of the average driveway?

The average driveway is 50 feet long, 15 feet wide, and one-quarter of a foot deep.

\[ vol\text{AvgDr} := \frac{50 \cdot 15 \cdot \frac{1}{4}}{4} \]

\[ vol\text{AvgDr} := \frac{375}{2} \]  \hspace{1cm} (13)

Part 2.b: Setup and enter the integral that computes the volume of a mixer of length \( L \)

\[ vol\text{Mixer} := \text{int}\left(\pi r^2, z = 0 \text{..} L\right) \]

\[ vol\text{Mixer} := \frac{49}{64} \frac{1}{4 L^2 + 1} \left( 16 L^3 + 16 L^2 \ln(4 L^2 + 1) + 4 L^2 \arctan(2 L) - 16 L^2 - 2 L \right. \]
\[ \left. + 4 \ln(4 L^2 + 1) + \arctan(2 L) \right) \]  \hspace{1cm} (14)

Part 2.c: Determine the length of the mixer needed to hold enough cement for an average driveway.

Set integral expression equal to average driveway volume and solve for \( L \).
solve: solve(volMixer = volAvgDr, L)
Warning, solutions may have been lost

Maple can't solve this symbolically, so we will use fsolve to solve it numerically.

> vol := fsolve(volMixer = volAvgDr, L)

vol := 13.53324525

We need a mixer of just over 13.5 feet to hold enough cement for an average driveway.

**Part 2.d: Compute the excess volume for each mixer that is longer than needed.**

The 10-foot mixer is too short.

Excess volume with a 15-foot mixer:

> vol15 := subs(L = 15, volMixer)

vol15 := 49/57664 \pi (50370 + 3604 \ln(901) + 901 \arctan(30))

> evalf(vol15 - volAvgDr)

16.1216552

Excess volume with a 20-foot mixer:

> vol20 := subs(L = 20, volMixer)

vol20 := 49/102464 \pi (121560 + 6404 \ln(1601) + 1601 \arctan(40))

> evalf(vol20 - volAvgDr)

69.8336075

**Problem 3: Gabriel's Horn**

> restart

**Part 3.a: Plot Gabriel's Horn**

> horn := 1/x

horn := 1/x

We'll use parametric equations similar to Problem 2.

> plot3d([x, horn*cos(t), horn*sin(t)], x = 1 .. 10, t = 0 .. 2*Pi, scaling = constrained)
Part 3.b: Compute the surface area of Gabriel's Horn

\[
\text{integrand} := 2 \cdot \pi \cdot \text{horn} \cdot \sqrt{1 + (\text{diff}(\text{horn}, x))^2}
\]

\[
\frac{2 \pi}{x} \sqrt{1 + \frac{1}{x^4}}
\]

\[
\text{integrand} := \frac{2 \pi}{x} \sqrt{1 + \frac{1}{x^4}}
\]

\[
> \text{surfaceArea} := \text{int}(\text{integrand}, x = 1 .. \infty)
\]

\[
\text{surfaceArea} := \infty
\]

(21)

Part 3.c: Compute the volume of Gabriel's Horn

\[
\text{volume} := \text{int}(\pi \cdot \text{horn}^2, x = 1 .. \infty)
\]

\[
\text{volume} := \pi
\]

(22)

Part 3.d: What is interesting about Gabriel's Horn?
Since the volume is finite, Gabriel's Horn could hold a finite amount of paint. However, the surface area is infinite, so it could not hold enough paint to cover its own surface!

Problem 4: Cylinder in a cone

\[
> \text{restart}
\]
Part 4.a: Redo problem 1, but now as a cylinder of radius \( w \) and height \( h \) fitting in a cone.

Equation for the line for the hypotenuse of the triangle

\[
> \text{line} := y = - \frac{H}{B} \cdot x + H
\]

Express \( h \) in terms of \( w \)

\[
> \text{hInTermsOfW} := \text{subs}(x = w, y = h, \text{line})
\]

General formula for the volume of a cylinder of radius \( r \) and height \( h \)

\[
> \text{volCylinderGeneral} := \pi r^2 \cdot h
\]

Sub in to get an equation just in terms of \( w \)

\[
> \text{volCylinder} := \text{subs}(r = w, \text{hInTermsOfW}, \text{volCylinderGeneral})
\]

To find the maximum volume, differentiate the volume, set the derivative equal to 0, and solve for \( w \)

\[
> \text{deriv} := \text{diff}(\text{volCylinder}, w)
\]

\[
> \text{optima} := \text{solve}(\text{deriv} = 0, w)
\]

The first value, 0, is clearly a minimum, not a maximum.

Radius of the cylinder of maximum volume:

\[
> w\text{Max} := \text{optima}[2]
\]

Height of the cylinder of maximum volume:

\[
> h\text{Max} := \text{subs}(w = w\text{Max}, \text{rhs(hInTermsOfW)})
\]

Volume of the cylinder of maximum volume:

\[
> \text{volCylinderMax} := \text{subs}(w = w\text{Max}, \text{volCylinder})
\]

Part 4.b: What is the ratio of the volume of the cylinder to the volume of the cone?

General formula for the volume of a cone of radius \( r \) and height \( h \)

\[
> \text{volConeGeneral} := \frac{\pi r^2 \cdot h}{3}
\]
\[ \text{volConeGeneral} := \frac{1}{3} \pi r^2 h \]  
(33)

Sub in to find volume of this cone.

\[ \text{volCone} := \text{subs}(r = B, h = H, \text{volConeGeneral}) \]
\[ \text{volCone} := \frac{1}{3} \pi B^2 H \]  
(34)

Ratio of volume of the cylinder to volume of the code

\[ \text{volRatio} := \frac{\text{volCylinderMax}}{\text{volCone}} \]
\[ \text{volRatio} := \frac{4}{9} \]  
(35)

The ratio for the rectangle in a triangle was 1/2. This ratio is just under 1/2, which does seem reasonable.