1. By playing with small values of $n$, can you discover and then prove a formula for
\[ \sum_{i=1}^{n} i = ? \]

2. 2.1. Can you give a rigorous argument why a formula for the sum of the first $n$ even numbers
\[ \sum_{i=1}^{n} (2i) = ? \]
follows from your first problem?

2.2. Derive from that the following identity
\[ \sum_{i=1}^{n} (2i - 1) = 1 + 3 + \ldots + (2n - 1) = n^2. \]

2.3. Can you discover and then prove by induction a formula for
\[ \sum_{i=0}^{n} i^2 = ? \]

2.4. Prove by induction on $n$ that
\[ \sum_{i=0}^{n} i^3 = \left( \sum_{i=0}^{n} i \right)^2. \]

3. (extra credit) As you perhaps observed, for the sum $\sum_{i=0}^{n} i$ the leading terms is $O(n^2)$, and for $\sum_{i=0}^{n} i^2$ the leading terms is $O(n^3)$. What’s the leading term in $\sum_{i=0}^{n} i^3$? Do you see a pattern? Can you prove something inductively for this? For any fixed $k$, can you find a relation with the integral
\[ \int_{0}^{n} x^k dx? \]

(Hint: Try to provide an upper bound as well as a lower bound relationship among the sum $\sum_{i=0}^{n-1} i^k$, $\int_{0}^{n} x^k dx$, and $\sum_{i=1}^{n} i^k$.)

4. Suppose $P(n)$ is a logical statement which is either true or false, for every integer $n \geq 0$.

4.1 Suppose $P(0)$ is true. And for all integers $n \geq 0$, if $P(n)$ is true, then so is $P(n+3)$. For what $n$, can you be certain that $P(n)$ is true?

4.2. Suppose $P(0)$, $P(1)$ and $P(2)$ are all true. And for all integers $n \geq 0$, if $P(n)$, $P(n+1)$ and $P(n+2)$ are true then so is $P(n+3)$. For what $n$, can you be certain that $P(n)$ is true?
4.3. Suppose $P(1)$ is true. And for all integer $n$, if $P(n)$ is true, then so is $P(2n)$. For what $n$, can you be certain that $P(n)$ is true?

4.4. Suppose $P(0)$ and $P(1)$ are true. And for all integers $n \geq 1$, if $P(n)$ is true, then so is $P(n + 2)$ and $P(n + 3)$. For what $n$, can you be certain that $P(n)$ is true?

Prove your answers.

5. A domino consists of two adjacent squares. Each square is 1 by 1, and a domino is 1 by 2.

We have a rectangle $R$ which is 2 by $n$, consisting of $2n$ squares.

We want to count how many ways $W(n)$ we can tile $R$ by $n$ dominos. Clearly when $n = 1$ there is exactly one way, $W(1) = 1$. and when $n = 2$ there are exactly two ways $W(2) = 2$.

Find a recursive relation for $W(n)$.

(We will assume the given 2 by $n$ rectangle has $2n$ squares that are labeled with integers 1 to $2n$. Imagine $R$ is placed horizontally in front of you, with two rows of squares, the northwest corner is numbered 1, the square just below it on the first column is numbered 2, etc. The southeast corner square is numbered $2n$. A tiling is a partition of the integer set $\{1, 2, \ldots, 2n\}$ into $n$ subsets, such that each subset corresponds to two adjacent squares.)

6. Design a DFA accepting all strings over $\{a, b\}$ such that the number of $a$’s plus twice the number of $b$’s is even.

7. Design a DFA accepting all strings over $\{a, b\}$ that constains $abab$ as a substring.

8. (extra credit) You are given 12 coins, and you are told that exactly 11 of which are real and one is fake. The only way to tell which is real and which is fake is by weighing them. All real coins weigh exactly the same, but the fake one weighs either heavier or lighter than the real one.

You are given a scale on which you can weigh any two disjoint subsets of the 12 coins. This is the basic operation: To weigh any chosen subset against another disjoint subset. For example, you can decide to weigh any 2 coins against another 2 coins. (There is no point to weigh 2 subsets of different cardinality. Do you see why?) Upon given the result (either the first subset weighs more or equal or less than the second one), your subsequent choice as to what to weigh can depend on this result.

Devise a strategy that identifies the fake coin in no more than 3 sequential weighings. Can your method tell whether the fake coin weighs more or less than the real one?

Note:

Please be concise. You are strongly advised to start thinking about the problems right away, and not wait till the due days.