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GREEDOIDS AND LINEAR OBJECTIVE FUNCTIONS*

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Abstract. Greedoids were introduced by the authors as generalizations of matroids providing a framework for the greedy algorithm. They can be characterized algorithmically via the optimality of the greedy algorithm for a class of objective functions, which are in general not linear and do not include all linear functions. It is therefore natural to ask the following questions: (1) What are those linear objective functions which can be optimized over any greedoid by the greedy algorithm; (2) what are those greedoids over which the linear objective function can be optimized by the greedy algorithm. This paper gives an answer to both questions. Moreover, it gives slimming procedures for obtaining such greedoids from matroids and it gives briefly some (negative) oracle results about greedoid optimization and greedoid recognition.

1. Introduction. In previous papers (Korte and Lovász [1981] and [1982a]) we have introduced greedoids as generalizations of matroids providing a framework for the greedy algorithm. Matroids can be characterized axiomatically as those subclusive set-systems for which the greedy solution is optimal for certain optimization problems (e.g. linear objective functions, bottleneck functions). Greedoids can also be characterized algorithmically via the optimality of the greedy algorithm for a class of objective functions, which are in general not linear and do not include all linear functions (cf. Korte and Lovász [1982a]). It is therefore natural to ask the following questions: (1) What are those linear objective functions which can be optimized over any greedoid by the greedy algorithm; (2) what are those greedoids over which any linear objective function can be optimized by the greedy algorithm. This paper gives an answer to both questions.

The algorithmic principle of greediness, i.e. of a locally myopic strategy, can be defined in different ways. The most common greedy approach is that of *best-in greedy*: starting with the empty set, the greedy solution will be built up recursively by adding the best possible element to it at each step, while remaining feasible. Another approach is that of *worst-out greedy*. Here we start with the complete ground set and eliminate from it in each step the worst-possible element as long as the remaining set is spanning. For matroids both approaches are equivalent, since the worst-out greedy is the best-in greedy for the negative objective function over the dual matroid. In the case of greedoids, it turns out that for general linear objective functions the worst-out greedy is optimal for a broader class of greedoids than the best-in-approach.

In § 2 we give some definitions and basic facts about greedoids, which will be needed in the rest of the paper. However, the interested reader is referred to Korte and Lovász [1982a] and [1982b] for a more detailed study of structural aspects of greedoids. Section 3 gives a compatibility characterization of linear objective functions which is sufficient to optimize these functions over *any* greedoid by the greedy algorithm. Section 4 characterizes those greedoids over which *any* linear objective function can be optimized by the worst-out greedy algorithm. A proper subclass of

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these greedoids has the property that the best-in greedy is optimal for any linear objective function. Section 5 gives some construction principles to obtain those greedoids by slimming a given matroid. Finally, in § 6 we state some (negative) oracle results about greedoids, among which it is noteworthy that the problem of optimizing an arbitrary linear objective function over a general greedoid given by a feasibility oracle is NP-hard. There is also no polynomial feasibility oracle algorithm to distinguish a greedoid from a matroid.

2. Definitions and basic facts about greedoids. We assume that the reader is familiar with the basic facts of matroid theory (cf. Welsh [1976]) and in general our notation is in accordance with the standard matroid terminology.

A *set-system* over a finite ground set E is a pair $(E, \mathcal{F})$ with $\mathcal{F} \subseteq 2^E$. A set-system is a *matroid* if the following axioms hold:

(M1) $\emptyset \in E$;

(M2) $X \subseteq Y \in \mathcal{F}$ implies $X \in \mathcal{F}$;

(M3) if $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there exists a $x \in Y - X$ such that $Y \cup \{x\} \in \mathcal{F}$.

A set-system which satisfies only (M1) and (M2) has little structure, but very different names. It is called *independence system*, *simplicial complex*, *subclusive* or *hereditary set-system*. For an arbitrary set-system $(E, \mathcal{F})$ we define its *hereditary closure* $\mathcal{H}$ as:

$$\mathcal{H} := \{X \subseteq Y \colon Y \in \mathcal{F}\}.$$

Another but more structural way to relax matroids is to keep the exchange axiom (M3) (and the trivial axiom (M1)) but to remove subclusiveness (M2); and this is exactly one way to define *greedoids*. There is another equivalent and even more natural way to define greedoids via extending the matroidal structure to languages, i.e. systems of ordered sets or strings, but for the purpose of this paper it is sufficient to consider only the unordered version of greedoid definition. Thus, we call a set-system $(E, \mathcal{F})$ a *greedoid* if (M1) and (M3) holds. (M1) and (M3) imply a weak subclusiveness, which we call *accessibility*:

(M2′) for all $X \in \mathcal{F}$ there exists $x \in X$ such that $X - \{x\} \in \mathcal{F}$.

Analogously to hereditary set-systems we call a set system which satisfies (M1) and (M2′) an *accessible set-system*. We define the *accessible kernel* $\mathcal{K}$ of a set-system $(E, \mathcal{F})$ as

$$\mathcal{K} := \{X \in \mathcal{F} \colon X = \{x_1, \cdots, x_k\} \text{ and } \{x_1, \cdots, x_i\} \in \mathcal{F} \text{ for all } 1 \leqq i \leqq k\}.$$

(M1) and (M3) are equivalent to (M1), (M2′), and

(M3′) if $X, Y \in \mathcal{F}$ and $|X| = |Y| + 1$, then there exists a $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

In the case of matroids (M3) and (M3′) are equivalently used, but this is only possible since (M2) holds. In analogy to matroid theory, we call sets which belong to $\mathcal{F}$ *feasible* or *independent*. Maximal independent sets are called *bases*. An element $d \in E$ is called *dummy*, if it does not occur in any feasible set. A greedoid is *normal*, if it has no dummy elements; it is called *full* if $E \in \mathcal{F}$.

For a greedoid we can define the (*independence*) *rank* of a set $X \subseteq E$ as:

$$r(X) := \max \{|A| \colon A \subseteq X, A \in \mathcal{F}\}.$$

This function has the following properties for $X, Y \subseteq E$ and $x, y \in E$

(R1) $r(\emptyset)=0$;

(R2) $r(X) \leqq |X|$;

(R3) if $X \subseteq Y$ then $r(X) \leqq r(Y)$;

(R4) if $r(X)=r(X\cup\{x\})=r(X\cup\{y\})$ then $r(X)=r(X\cup\{x\}\cup\{y\})$.

Conversely, a function $r: 2^E \to \mathbb{Z}$ satisfying (R1), (R2). (R3) and (R4) defines uniquely a greedoid (cf. Korte and Lovász [1982a]). These axioms are again direct relaxations of the rank definition of matroids, which in addition have the *unit increase property*:

$$r(X\cup\{x\}) \leqq r(X)+1 \quad \text{for } X \subseteq E,\ x \in E.$$

From (R1)–(R4) and the unit increase property one derives in matroid theory that the rank function is *submodular*, i.e. $r(X\cap Y)+r(X\cup Y) \leqq r(X)+r(Y)$. This fails to hold for greedoids in general; but the property (R4), which we call *local submodularity*, is often a reasonable substitute.

In contrast to matroids, the intersection of a set with a basis of a greedoid may have larger cardinality than the rank of this set. Therefore we define the *basis rank* of a set $X \subseteq E$ as

$$\beta(X) := \max\{|X\cap B|: B \in \mathcal{F}\}.$$

Clearly, $\beta(X) \geqq r(X)$. A set $X \subseteq E$ is called *rank-feasible* if $\beta(X)=r(X)$. We denote the family of all rank feasible sets by $\mathcal{R}=\mathcal{R}(E,\mathcal{F})$. Clearly, $\mathcal{F} \subseteq \mathcal{R}$ and $\mathcal{F}=\mathcal{R}$ for a full greedoid. In general $(E,\mathcal{R})$ is not a greedoid and $\mathcal{R}$ is not closed under union.

We recall here some facts about rank-feasibility (cf. Korte and Lovász [1982b]): A greedoid is a matroid iff $\mathcal{R}=2^E$. For $A, B \subseteq E$ we have

$$\beta(A\cup B)+r(A\cap B) \leqq \beta(A)+\beta(B)$$

and consequently, if $A, B \in \mathcal{R}$ then

$$r(A\cup B)+r(A\cap B) \leqq r(A)+r(B),$$

i.e. r is submodular. This can be also derived from the fact that $A \in \mathcal{R}$ iff $r(A\cup X) \leqq r(A)+|X|$ for all $X \subseteq E-A$.

A fundamental concept in matroid theory is the closure operator. Therefore we define analogously for greedoids the (*rank*) *closure* of a set $X \subseteq E$ as

$$\sigma(X) := \{x \in E: r(X\cup\{x\})=r(X)\}.$$

This operator is not monotone, but it has the following properties:

(C1) $X \subseteq \sigma(X)$ for all $X \subseteq E$;

(C2) if $X \subseteq Y \subseteq \sigma(X)$ then $\sigma(X)=\sigma(Y)$;

(C3) if $X \subseteq E$ and $x \in E-X$ such that for all $z \in X\cup x$, $z \notin \sigma(X\cup x-z)$, and $x \in \sigma(X\cup y)$, then $y \in \sigma(X\cup x)$.

It was shown in Korte and Lovász [1982a] that a mapping $\sigma: 2^E \to 2^E$ satisfying (C1), (C2), and (C3) uniquely defines a greedoid.

The closure axioms for greedoids are again relaxations of the closure for matroids. (C1) is trivial, (C2) follows from monotonicity and idempotence, and (C3) is a weakening of the Steinitz–McLane axiom for matroids. It can be shown that (C2) implies idempotence, but of course not monotonicity.

A set $X \subseteq E$ is called *closed* if $X = \sigma(X)$. An easy construction leads to a *monotone* closure operator, namely

$$\mu(X) := \bigcap \{Y : X \subseteq Y \text{ and } Y \text{ closed}\}.$$

μ does not determine the greedoid uniquely. In fact, for a full greedoid we have $\mu = \mathrm{id}$.

We call a set *closure-feasible* if $X \subseteq \sigma(A)$ implies $X \subseteq \mu(A)$, or—which is equivalent—if $X \subseteq \sigma(A)$ implies $X \subseteq \sigma(B)$ for $A \subseteq B \subseteq E$. The family of all closure feasible sets will be denoted by $\mathscr{C} = \mathscr{C}(E, \mathscr{F})$. The family $\mathscr{C}$ is closed under union and we have $\mathscr{C} \subseteq \mathscr{R}$. Further, $\mathscr{C}$ with inclusion as a partial order forms a lattice with the operation $A \vee B := A \cup B$ and $A \wedge B := \bigcup \{C \in \mathscr{C} : C \subseteq A \cap B\}$. The rank function r is submodular on this lattice. $(E, \mathscr{C})$ is not a greedoid in general, but the accessible kernel $\mathscr{K} = \mathscr{K}(\mathscr{C})$ of $\mathscr{C}$ defines trivially a greedoid. The rank function does not have the unit increase property on $\mathscr{C}$. But since $\mathscr{K} \subseteq \mathscr{C}$ is also a lattice, the rank function is also submodular on $\mathscr{K}$.

A very substantial subclass of greedoids are *interval greedoids*. We call a greedoid $(E, \mathscr{F})$ an interval greedoid if for all $A, B, C \in \mathscr{F}$ with $A \subseteq B \subseteq C$ and $x \in E - C$ such that $A \cup x \in \mathscr{F}$ and $C \cup x \in \mathscr{F}$, it follows that $B \cup x \in \mathscr{F}$. In Korte and Lovász [1982b] it was shown that a greedoid is an interval greedoid iff $\mathscr{C} = \mathscr{R}$ and iff $\mathscr{F} \subseteq \mathscr{R}$. Generally, no inclusion relation holds between $\mathscr{F}$ and $\mathscr{C}$. Furthermore, if $(E, \mathscr{F})$ is an interval greedoid, then already $(E, \mathscr{C})$ is a greedoid. We call a normal greedoid a *shelling structure* if the *interval property* mentioned above holds *without upper bounds*, i.e. if for all $A \subseteq B$ and $x \in E - B$ such that $A \cup x \in \mathscr{F}$ it follows $B \cup x \in \mathscr{F}$. Shelling structures are studied in greater detail in Korte and Lovász [1983a].

3. Special linear objective functions and general greedoids. An optimization problem over a greedoid $(E, \mathscr{F})$ can be described by introducing a linear objective function $w : E \to \mathbb{R}$ as a weighting of the elements of the ground set. This function can be extended to a modular function $w : 2^E \to \mathbb{R}$ by $w(X) := \sum_{x \in X} w(x)$ for all $X \subseteq E$. For reasons of simplicity we will consider in the following only maximization problems, i.e.

$$\max \{w(F) : F \in \mathscr{F}\}.$$

We call a basis X of $(E, \mathscr{F})$ an *optimal basis* for which $w(X)$ is maximal among all bases.

The principle of the *greedy algorithm* (or more precisely: the *best-in greedy algorithm*) can be briefly described by the greedy bases, which are obtained with this algorithm. We call a basis $\{x_1, \cdots, x_r\}$ of a greedoid $(E, \mathscr{F})$ a (*best-in*) *basis for* w if it is obtained by the following recurrence: x_{i+1} is the element with the largest weight in $E - \{x_1, \cdots, x_i\}$ such that $\{x_1, \cdots, x_i, x_{i+1}\} \in \mathscr{F}$.

In the next section we refer to a *worst-out greedy algorithm* which in contrast starts with the ground set E and eliminates elements with the smallest possible weight as long as the remaining set is spanning, i.e. contains a basis. The *worst-out greedy basis for* w is then a basis $Y = E - \{x_1, \cdots, x_k\}$ which is obtained by the recurrence: x_{i+1} is the element with smallest weight in $E - \{x_1, \cdots, x_i\}$ such that $E - \{x_1, \cdots, x_i, x_{i+1}\}$ is spanning. It is an easy observation that for matroids the best-in greedy basis and the worst-out greedy basis are identical.

In general, an arbitrary linear objective function cannot be optimized over a greedoid with the greedy algorithm. Therefore, we need the following compatibility definition: Let $\mathscr{S} \subseteq 2^E$, and let $w : E \to \mathbb{R}$. We say that w is $\mathscr{S}$*-compatible* if $\{x \in E : w(x) \geqq c\} \in \mathscr{S}$ for all $c \in \mathbb{R}$, i.e. all *level sets* of w are in $\mathscr{S}$. As usual, we call a function $w : E \to \{0, 1\}$ the *characteristic function* of a set $X \subseteq E$ iff $w(x) = 1$ for all $x \in X$.

Then the definition of rank-feasibility implies immediately the following:

LEMMA 3.1. *If w is the characteristic function of a rank-feasible set, then all greedy bases are optimal.*

Our aim is to prove the following theorem:

THEOREM 3.2. *Let $(E, \mathcal{F})$ be a greedoid and $w: E \to \mathbb{R}$ be an $\mathcal{R}$-compatible weighting. Then all greedy basis for w are optimal.*

Proof. We can write w in the form

$$w = \sum_{i=1}^{t} \lambda_i w_i,$$

where $w_1 \leqq w_2 \leqq \cdots \leqq w_t$ are characteristic functions of rank-feasible sets, and $\lambda_1, \cdots, \lambda_t > 0$. In fact, let $c_1 > c_2 > \cdots > c_t$ be the different values assumed by w over 2^E, and let X_i be the level set $X_i = \{x: w(x) \geqq c_i\}$. Then we can choose w_i to be the characteristic function of X_i and $\lambda_i = c_i - c_{i+1}$, $\lambda_t = c_t$.

Let X be a greedy basis for w. Then X is, clearly, a greedy basis for each w_i. So by Lemma 3.1, X is an optimal basis for each w_i. But then, clearly, X is an optimal basis for w. $\square$

Remark. Faigle [1979] considers certain accessible set-systems called *generating systems* and proves that the best-in greedy algorithm optimizes certain linear objective functions over them. While his systems are not necessarily greedoids, those feasible subsets of his "generating systems" which come up in a greedy basis do form a greedoid. Based on this, it is easy to derive Faigle's result from Theorem 3.2. For a more detailed discussion of the relationship between greedoids and Faigle's structures, see Korte and Lovász [1983b].

4. Special greedoids and general linear objective functions. We now invert the question of the last section and ask how much we have to restrict greedoids such that a greedy basis for any arbitrary linear objective function is optimal. The next theorem gives necessary and sufficient conditions for the worst-out greedy.

THEOREM 4.1. *For a greedoid $(E, \mathcal{F})$ the following statements are equivalent:*

(1) *Let B_1, B_2 be bases of $(E, \mathcal{F})$; for every $x \in B_1 - B_2$ there exists a $y \in B_2 - B_1$ such that $B_2 \cup x - y \in \mathcal{F}$.*
(2) *The hereditary closure $\mathcal{M}$ of $\mathcal{F}$ is a matroid $(E, \mathcal{M})$.*
(3) *β is submodular.*
(4) *For every linear objective function w a worst-out greedy basis is optimal.*

Proof. (1)$\Leftrightarrow$(2) is known from matroid theory.

(2)$\Rightarrow$(3): It suffices to show that β is the rank function of $(E, \mathcal{M})$. Let $X \subseteq E$; then

$$\beta(X) = \max\{|B \cap X|: B \in \mathcal{F}\} = \max\{|U|: U \subseteq X, U \in \mathcal{M}\},$$

since $\mathcal{M}$ is the hereditary closure.

(2)$\Rightarrow$(4): The spanning sets for $\mathcal{F}$ and $\mathcal{M}$ are the same, and so the worst-out greedy basis for $\mathcal{F}$ and $\mathcal{M}$ are the same. We know from matroid theory that the worst-out greedy bases are optimal for $\mathcal{M}$.

(3)$\Rightarrow$(2): Trivially, β has the unit increase property. Hence β is a matroid rank function. But $X \in \mathcal{M}$ iff $\beta(X) = |X|$. So $(E, \mathcal{M})$ is the matroid determined by β.

(4)$\Rightarrow$(2): Let $\mathcal{M}^* := \{X \subseteq E$: there exists a basis B with $B \cap X = \emptyset\}$. Then a worst-out greedy bases for $\mathcal{F}$ is optimal iff a best-in greedy basis for $\mathcal{M}^*$ is optimal. But this is the case iff $(E, \mathcal{M}^*)$ is a matroid which is equivalent to the fact that $(E, \mathcal{M})$ is a matroid. $\square$

Remarks. 1. Condition (1) is not enough to guarantee the optimality of the best-in greedy: Let $E=\{a, b, c\}$ and $\mathcal{F}=\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The greedoid $(E, \mathcal{F})$ satisfies (1). However, with $w(a)=1$, $w(b)=0$, $w(c)=M \gg 1$, the best-in greedy basis is $\{a, b\}$ with weight 1, while the optimal basis is $\{b, c\}$ with weight M.

2. Let $\sigma_{\mathcal{M}}$ denote the matroid closure; then we have for greedoids with condition (1) that $\sigma_{\mathcal{M}}(A)=\mu(A)$ for $A \in \mathcal{F}$. In fact, $y \in \mu(A)$ iff $y \in A$ or $A \subseteq B$, $B \in \mathcal{F}$ implies $y \notin B$ which is equivalent to $y \in \sigma_{\mathcal{M}}(A)$.

The following theorem gives optimality conditions for the best-in greedy, which are of the same kind, but more restrictive.

THEOREM 4.2. *For a greedoid $(E, \mathcal{F})$ the following statements are equivalent*:

(1) *Let $A \in \mathcal{F}$, $B \supseteq A$, be a basis of $(E, \mathcal{F})$ and let $x \in E-B$ and $A \cup \{x\} \in \mathcal{F}$. Then there exists a $y \in B-A$ with $A \cup y \in \mathcal{F}$ such that $B \cup x - y \in \mathcal{F}$ (strong exchange property).*

(2) *The hereditary closure $\mathcal{M}$ of $\mathcal{F}$ is a matroid $(E, \mathcal{M})$ and every set which is closed in $\mathcal{F}$ ($\mathcal{F}$-closed) is also closed in $\mathcal{M}$ ($\mathcal{M}$-closed).*

(3) *For every linear objective function w a (best-in) greedy basis is optimal.*

Proof. (1)$\Rightarrow$(3). Let $w: E \to \mathbb{R}$ be any objective function, B an optimum basis, and $a_1, a_2, \cdots, a_r$ a best-in greedy basis, chosen in this order. Let $a_1, \cdots, a_k \in B$, but $a_{k+1} \notin B$ and choose B so that k is maximal. Let $A := \{a_1, \cdots, a_k\}$. By (1), there exists a $y \in B-A$ such that $A \cup y \in \mathcal{F}$ and $B \cup a_{k+1} - y \in \mathcal{F}$. By greediness, $w(y) \leqq w(a_{k+1})$ and so $w(B \cup a_{k+1} - y) \geqq w(B)$. Since B is optimal, we have $w(B \cup a_{k+1} - y) = w(B)$ and so $B \cup a_{k+1} - y$ is also optimal, which contradicts the maximality of k.

(3)$\Rightarrow$(2). Let $X, Y \in \mathcal{M}$, $|X|<|Y|$. Let $0<t<1$ and define $w(x)=1$ if $x \in X$, $w(x)=t$ if $X \in Y-X$, and $w(x)=0$ otherwise. The set of greedy bases is independent of the value of t. If $t \approx 0$, then every optimal basis must contain X. Hence every greedy basis must contain X. But if $t>|X-Y|/|Y-X|$ then there is a basis containing Y, and hence the maximal objective value is greater or equal to $t|Y-X|+|Y \cap X|>|X|$. So a greedy basis must contain some element $y \in Y-X$ (besides X). Then $X \cup y \in \mathcal{M}$.

Thus, we know that $(E, \mathcal{M})$ is a matroid. It remains to show that every $\mathcal{F}$-closed set is also $\mathcal{M}$-closed. Let U be any $\mathcal{F}$-closed set, A an $\mathcal{F}$-basis of U, and extend A to an $\mathcal{M}$-basis A' of U. Let $v \in E-U$. Consider the objective function $w(x)=1$ if $x \in A'$ and $w(x)=0$ otherwise. Then there exists a basis containing A', and so every optimal basis contains A'. Of course, every best-in greedy basis also contains A'. But there must be also a greedy basis B starting with $A \cup v$, and so $(A \cup v) \cup A' = A' \cup v \subseteq B$. Thus $A' \cup v \in \mathcal{M}$ and so $v \notin \sigma_{\mathcal{M}}(A')$, ($\mathcal{M}$-closure of A'). This holds for all $v \in E-U$, so $\sigma_{\mathcal{M}}(A') \subseteq U$. But A' is an $\mathcal{M}$-basis of U, so $\sigma_{\mathcal{M}}(A')=U$ and so U is $\mathcal{M}$-closed.

(2)$\Rightarrow$(1). Consider $\sigma_{\mathcal{F}}(A)$, ($\mathcal{F}$-closure of A); by hypothesis $\sigma_{\mathcal{F}}(A)$ is also $\mathcal{M}$-closed. $B \cup x$ has a unique (fundamental) $\mathcal{M}$-circuit C. We have $x \in C-\sigma_{\mathcal{F}}(A)$, but since $\sigma_{\mathcal{F}}(A)$ is $\mathcal{M}$-closed, it follows that $|C-\sigma_{\mathcal{F}}(A)| \geqq 2$. Let $y \in C-\sigma_{\mathcal{F}}(A)-x$. Then $A \cup y \in \mathcal{F}$ and $B \cup x - y \in \mathcal{M}$, but $B \cup x - y$ is a basis of $\mathcal{M}$, and so a basis of $\mathcal{F}$. □

Remark. Condition (1) of Theorem 4.2 was independently observed by Goetschel [1983].

5. Slimmed matroids. It is a natural question to ask what greedoids satisfy the conditions of Theorems 4.2 and 4.1. Of course, matroids and trivially also all full greedoids do so. A nontrivial class are undirected branching greedoids. In Korte and Lovász [1982a] we have described a *search* or *directed branching greedoid* $(E, \mathcal{F})$ by a directed graph G and a root $r \in V(G)$. Let $E=E(G)$ and let $\mathcal{F}$ be the set of arc-sets of all arborescences in G rooted at r. The bases of $(E, \mathcal{F})$ are maximal branchings in G. In contrast, the *undirected branching greedoid* contains as feasible sets all cycle-free connected subgraphs of G which contain r. It is easy to see that this greedoid satisfies condition (2) of Theorem 4.2. (The directed branching greedoid does not.)

On the other hand the conditions of Theorem 4.1 give rise to general constructions of greedoids from a given matroid, whose set of bases is the same, but the feasible set is slimmed. In the following we will introduce some construction principles of slimming a matroid.

Given a matroid $(E, \mathcal{M})$ we call a greedoid $(E, \mathcal{F})$ a *slimming* of the matroid $(E, \mathcal{M})$ if $\mathcal{F} \subseteq \mathcal{M}$ and all bases of $\mathcal{M}$ remain bases of $\mathcal{F}$. The undirected branching greedoid is a slimming of the graphic matroid, actually an intersection of the graphic matroid with the *line search greedoid*, which is a shelling structure defined on the same graph G where $\mathcal{F}$ is the collection of all edge-sets which are connected and contain r (cf. Korte and Lovász [1982a]).

The next theorem describes the first slimming procedure.

THEOREM 5.1. *Let $(E, \mathcal{M})$ be a matroid with rank function $r_{\mathcal{M}}$ and $r_{\mathcal{M}}(E) = k$. Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{k-1} \subseteq E$ such that $r_{\mathcal{M}}(E - A_i) \leqq k - i$. Define*

$$\mathcal{F} := \{X \in \mathcal{M} : |X \cap A_i| \geqq i \text{ for } 1 \leqq i \leqq |X|\}.$$

Then $(E, \mathcal{F})$ is a greedoid and a slimming of $(E, \mathcal{M})$.

Proof. We first show that $(E, \mathcal{F})$ is a greedoid. To prove (M3′) we take $X, Y \in \mathcal{F}$ with $|X| = |Y| + 1$. Then there exists an $x \in X - Y$ such that $Y \cup x \in \mathcal{M}$. But $|Y \cap A_i| \geqq i$ and hence $|(Y \cup x) \cap A_i| \geqq i$ for $1 \leqq i \leqq |Y|$ as $Y \in \mathcal{F}$. Further, $|X \cap A_{|Y|+1}| \geqq |Y| + 1 = |X|$ since $X \in \mathcal{F}$ and so $X \subseteq A_{|Y|+1}$, in particular $x \in A_{|Y|+1}$. Hence

$$|(Y \cup x) \cap A_{|Y|+1}| \geqq 1 + |Y \cap A_{|Y|+1}| \geqq 1 + |Y \cap A_{|Y|}| \geqq 1 + |Y|.$$

So $Y \cup x \in \mathcal{F}$.

Further, $\mathcal{F}$ is accessible. For, let $X \in \mathcal{F}$, and let i be the least index such that $X \subseteq A_i$. Since $X \in \mathcal{F}$, we have $i \leqq |X|$. Let $x \in X \cap (A_i - A_{i-1})$. Then $X - x \in \mathcal{M}$ and $|(X - x) \cap A_j| = |X - x| \geqq j$ if $j \geqq i$ and $|(X - z) \cap A_j| = |X \cap A_j| \geqq |X| > |X - x|$ if $j < i$. Hence $X - x \in \mathcal{F}$.

It remains to prove that $\mathcal{F}$ contains all bases of $\mathcal{M}$. Let B be a basis of $\mathcal{M}$; then $|B \cap A_i| = k - |B \cap (E - A_i)| \geqq k - r_{\mathcal{M}}(E - A_i) \geqq i$. □

Remarks. 1. The rank function $r_{\mathcal{F}}$ of $\mathcal{F}$ can be obtained by the following formula:

$$r_{\mathcal{F}}(X) := \max \{i : r_{\mathcal{M}}(X \cap A_j) \geqq j \text{ for } 1 \leqq j \leqq i\}.$$

2. It can be easily verified that the family

$$\mathcal{F}_0 = \{X \subseteq E : |X \cap A_i| \geqq i \text{ for } 1 \leqq i \leqq |X|\}$$

defines a shelling structure. Hence $\mathcal{F} := \mathcal{M} \cap \mathcal{F}_0$ is the intersection of a matroid with a shelling structure, and therefore an interval greedoid.

Another slimming procedure is given by

THEOREM 5.2. *Let $(E, \mathcal{M})$ be a matroid, $(E, \mathcal{F})$ a greedoid and suppose that the following hold:*

(1) *For $X, Y \in \mathcal{M}$ such that $\sigma_{\mathcal{M}}(X) = \sigma_{\mathcal{M}}(Y)$, we have $X \in \mathcal{F}$ iff $Y \in \mathcal{F}$.*

(2) *All (or equivalently at least one) bases of $\mathcal{M}$ are in $\mathcal{F}$.*

Then $(E, \mathcal{M} \cap \mathcal{F})$ is a greedoid which is a slimming of $\mathcal{M}$.

Proof. We show (M3). Suppose $X, Y \in \mathcal{M} \cap \mathcal{F}$, $|X| > |Y|$. Extend $\sigma_{\mathcal{M}}(Y) \cap X$ to an $\mathcal{M}$-basis X_1 of $\sigma_{\mathcal{M}}(Y)$. Then $\sigma_{\mathcal{M}}(X_1) = \sigma_{\mathcal{M}}(Y)$ and so by (1), $X_1 \in \mathcal{F}$. Since $|X_1| < |X|$, there exists a $x \in X - X_1$ such that $X_1 \cup x \in \mathcal{F}$. But $x \notin \sigma_{\mathcal{M}}(Y) = \sigma_{\mathcal{M}}(X_1)$ since $X \cap \sigma_{\mathcal{M}}(Y) \subseteq X_1$, but $x \notin X_1$. Hence $X_1 \cup x \in \mathcal{M}$ and so $X_1 \cup x \in \mathcal{M} \cap \mathcal{F}$. But $\sigma_{\mathcal{M}}(X_1 \cup x) = \sigma_{\mathcal{M}}(Y \cup x)$ and so $Y \cup x \in \mathcal{M} \cap \mathcal{F}$ by (1). □

The next theorem gives a further slimming construction.

THEOREM 5.3. (a) *Let* $(E, \mathcal{M})$ *be a matroid,* $\mathcal{G}$ *an accessible family of flats in* $\mathcal{M}$, *closed under union in the geometric lattice of* $(E, \mathcal{M})$. *Let*

$$\mathcal{F}_0 := \{X \in \mathcal{M} : \sigma_{\mathcal{M}}(X) \in \mathcal{G}\}$$

and let $\mathcal{F}$ *be the accessible kernel of* $\mathcal{F}_0$. *Then* $(E, \mathcal{F})$ *is a greedoid.*

(b) *Moreover,* $(E, \mathcal{F})$ *is a slimming of* $(E, \mathcal{M})$ *iff the following holds: for every* $F \in \mathcal{G}$ *and* $F_1, \cdots, F_t \notin \mathcal{G}$ *such that* $F_1, \cdots, F_t$ *cover* F *in the lattice, we have that* $F_1 \cup \cdots \cup F_t$ *is nonspanning in* $(E, \mathcal{M})$.

Proof. (a) We show (M3): Let $X, Y \in \mathcal{F}$, $|X| > |Y|$. By accessibility, $X = \{x_1, \cdots, x_m\}$ such that $\{x_1, \cdots, x_i\} \in \mathcal{F}$ for all $1 \leqq i \leqq m$. Let i be the first index with $x_i \notin \sigma_{\mathcal{M}}(Y)$. Then $Y \cup x_i \in \mathcal{M}$. Furthermore $\sigma_{\mathcal{M}}(Y \cup x_i) = \sigma_{\mathcal{M}}(\sigma_{\mathcal{M}}(Y) \cup \sigma_{\mathcal{M}}(x_1, \cdots, x_i)) \in \mathcal{G}$. Hence $Y \cup x_i \in \mathcal{F}$.

(b) I. By accessibility of $\mathcal{F}$, there exists a sequence of flats $B_0 \subset B_1 \subset \cdots \subset B_m \in \mathcal{F}$ such that $B_i \in \mathcal{G}$ and $r(B_i) = i$. Let $b_i \in B_i - B_{i-1}$; then $\{b_1, \cdots, b_m\} \in \mathcal{F}$. If $F_1 \cup \cdots \cup F_t$ is spanning, we can extend $\{b_1, \cdots, b_m\}$ to a basis A of $(E, \mathcal{F})$. Let $a \in A - \{b_1, \cdots, b_m\}$ such that $\{b_1, \cdots, b_m, a\} \in \mathcal{F}$. Then $a \in F_v$ for some $1 \leqq v \leqq t$, and so $\sigma_{\mathcal{M}}(\{b_1, \cdots, b_m, a\}) = F_v$. But $\{b_1, \cdots, b_m, a\} \in \mathcal{F}$ implies $\sigma_{\mathcal{M}}(\{b_1, \cdots, b_m, a\}) \in \mathcal{G}$, a contradiction.

II. Let b be any basis of $\mathcal{M}$. Consider a maximal subset $A \subseteq B$ with $A \in \mathcal{F}$. We claim that $A = B$. Suppose not, and let $F = \sigma_{\mathcal{M}}(A)$, $B - A = \{b_1, \cdots, b_t\}$ and let $F_i = \sigma_{\mathcal{M}}(A \cup b_i)$. Then $\bigcup F_i$ is spanning in $(E, \mathcal{M})$, because $B \subseteq \bigcup F_i$. Thus, there exists an $F_i \in \mathcal{G}$. But then $A \cup b_i \in \mathcal{F}$, contradiction. □

Remark. If $(E, \mathcal{M})$ is the free matroid, then the construction of Theorem 5.3 gives every shelling structure $(E, \mathcal{F}_1)$ by letting $\mathcal{G} = \mathcal{F}_1$.

6. Oracle results. In this final section we mention briefly some negative results about greedoid optimization and greedoid recognition obtained by an oracle approach. We do not go into details of oracle techniques here. The reader is referred to similar approaches for independence systems and matroids in earlier papers (cf. Hausmann and Korte [1981] and Jensen and Korte [1982]). As in the case of matroids we assume that the greedoid $(E, \mathcal{F})$ is given by a *feasibility oracle*, i.e. a mapping $O: 2^E \to \{\text{Yes}, \text{No}\}$ which is defined for $X \subseteq E$ as $O(X) = \text{Yes}$ if $X \in \mathcal{F}$, $O(X) = \text{No}$ otherwise.

It is clear that a feasibility oracle uniquely determines the greedoid. Moreover, several questions concerning greedoids can be decided in polynomial time using the feasibility oracle: e.g. computing the rank or closure of a set, as well as the problems discussed in previous chapters. However, some other important questions cannot be decided by good algorithms. To formulate these negative results, we need the following definition.

A problem concerning greedoids given by a feasibility oracle is called NP-hard, if there is a special class of greedoids, with some "name" (encoding) for each member, such that the oracle can be realized by a polynomial-time algorithm for members of this class (polynomial in the length of the "name") and the problem is NP-hard already for members of this class.

THEOREM 6.1. *The problem of optimizing a linear objective function over the bases of an arbitrary greedoid given by a feasibility oracle is* NP*-hard.*

Proof. We consider the *k-truncation* of the directed or undirected branching greedoid $(E, \mathcal{F})$, i.e. the greedoid $(E, \mathcal{F}^{(k)})$ with $\mathcal{F}^{(k)} := \{X \subseteq E : X \in \mathcal{F} \text{ and } |X| \leqq k\}$. The problem of finding a maximum weighted branching of size less or equal to k includes the *Steiner problem*, which is known to be NP-hard. □

Remark. The problem of optimizing an arbitrary linear objective function over the feasible sets of a greedoid remains NP-hard even for shelling structures. In fact, this optimization problem for line search greedoids also contains the Steiner problem.

THEOREM 6.2. *There is no polynomial-time algorithm to decide whether a greedoid given by a feasibility oracle is a matroid.*

Proof. Consider the uniform matroid $(E, \mathcal{M})$ of rank $r = |E|/2$ and the greedoid $(E, \mathcal{F})$ with $\mathcal{F} := \mathcal{M} - \{X\}$ where $|X| = r-1$. With the usual argument (cf. Hausmann and Korte [1981]) one can show that any feasibility oracle algorithm can not distinguish between $(E, \mathcal{M})$ and $(E, \mathcal{F})$ using only polynomially many calls on the feasibility oracle. □

COROLLARY 6.3. *There is no polynomial algorithm to recognize a closure feasible set for a greedoid given by a feasibility oracle, i.e. to decide membership in* $\mathcal{C}$.

Proof. It is an easy observation that a greedoid $(E, \mathcal{F})$ is a matroid iff $\{x\} \in \mathcal{C}$ for all $x \in E$. (To prove this one needs that $\mathcal{C}$ is closed under union.) Then apply Theorem 6.2. □

THEOREM 6.4. *There is no polynomial-time algorithm to decide whether a greedoid given by a feasibility oracle is normal.*

Proof. Let $(E, \mathcal{F})$ be a uniform matroid of rank $r = |E|/2$. Let $d \notin E$ and consider the greedoid $(E \cup \{d\}, \mathcal{F})$. Let $X \subseteq E$, $|X| = r-1$ and $\mathcal{F}' = \mathcal{F} \cup \{X \cup \{d\}\}$. Then it is easy to check that $(E \cup \{d\}, \mathcal{F}')$ is also a greedoid. By the usual argument again, no feasibility oracle algorithm can distinguish between $(E \cup \{d\}, \mathcal{F})$ and $(E \cup \{d\}, \mathcal{F}')$ in polynomial time. □

COROLLARY 6.5. *There is no polynomial-time algorithm to decide whether a given element is a dummy.*

COROLLARY 6.6. *There is no polynomial-time algorithm to recognize a rank-feasible set in a greedoid given by a feasibility oracle.*

Proof. Observe that $d \in E$ is a dummy iff $\{d\} \in \mathcal{R} - \mathcal{F}$.

THEOREM 6.7. *It is* NP-*hard to recognize for a greedoid a rank-feasible* (*or closure-feasible*) *set, i.e. to decide membership in* $\mathcal{R}$ (*or in* C).

Proof. Let G be a digraph, $E = E(G)$, $V(G) = \{v_1, \cdots, v_n\}$. We call an arc e a *shortcut* in G if there exists a dipath in $G - e$ from the tail of the head of e. Let

$$\mathcal{F} := \{e_1, \cdots, e_k : e_i \text{ is not a shortcut in } G - \{e_1, \cdots, e_{i-1}\}\}.$$

Then $(E, \mathcal{F})$ is a shelling structure, which we call the *digraph shortcut greedoid.* This greedoid was first observed by A. Björner [1983]. It can be also represented as a *convex shelling structure* (cf. Korte and Lovász [1983a]) in $\mathbb{R}^n$ of the following set of points $\{0, e_{ij}\}$ where 0 is the 0-vector and e_{ij} is a 0, ± 1 incidence vector of the arc $e = (v_i, v_j)$ which has a -1 at the ith component, a $+1$ at the jth component and 0's elsewhere. Then $\{0\} \in \mathcal{F}$ iff G is acyclic. We take the k-truncation of this greedoid. Then $\{0\} \notin \mathcal{R}$ iff the feedback number of G is $\leqq k-1$, but this is a well-known NP-hard problem. □

This shortcut greedoid is an interval greedoid, and thus $\mathcal{R} = \mathcal{C}$. So the assertion concerning closure feasibility follows in the same way.

Remark. The test for membership in $\mathcal{R}$ is of course a special case of optimizing a linear objective function over $(E, \mathcal{F})$.

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