CS 787
Spring 2023
Homework #2

Due in class Wednesday, April 26, 2023

Rules for Homework. See Homework 1.

1. Let $G = (V, E)$ be a directed graph with two distinguished vertices: a source s and a sink t. According to some authors,

   i) A flow is a non-negative function on edges, satisfying the Kirchoff current law at all $v \neq s, t$.

   According to some others,

   ii) A flow is a skew-symmetric function on $V \times V$ satisfying the Kirchoff current law at all $v \neq s, t$.

   For this problem you will investigate how these definitions differ.

   a) Suppose that G does not contain any subgraph of the form
   $$v \underset{\leftarrow}{\overset{\rightarrow}{}} w.$$
   Show the two definitions are equivalent, in following sense: a flow according to one definition naturally determines a flow according to the other.

   b) Consider the unit capacity network with flow values indicated below:
   $$s \overset{x}{\rightarrow} a \underset{w}{\overset{z}{\rightleftarrows}} b \overset{x}{\rightarrow} t$$
   Find necessary and sufficient conditions on z and w for this to be a flow in the sense of ii).

   c) Starting with $x = z = w = 0$ in the network of b), push 1 unit along $s \to a \to b \to t$. Comment on whether this is a flow according to i) or ii).

2. A baseball league has n teams, each of whom will play a games against each other team.

   a) How many games does each team play? How many games total are there in a season?

   b) On opening day, hope springs eternal as it is possible that any particular team (team 1, say), could win. Use the flow network model we have discussed for baseball elimination to verify this.

   c) Check the result of b) by giving a proof that does not rely on flow networks. (It should be intelligible to any sports fan.)

d)* Are there conditions under which the league leader is eliminated? There are bizarre schedules under which this is possible. For example, suppose A is scheduled to play only 2 games and then beats B and C. If B and C are scheduled to play each other 7 times, it is certain that one of them will get more wins than A.

3. Consider minimax assignment problems where the rows are nondecreasing. That is, if the $n \times n$ cost matrix is (c_{ij}), then for each i, we have $c_{i1} \leq c_{i2} \leq \cdots \leq c_{in}$. We seek a permutation σ of $1..n$ that minimizes $\max_i\{c_{i,\sigma(i)}\}$.
   a) Give an integer programming formulation for this problem.
   b) Give an algorithm to solve it that runs in polynomial time.
   c) Prove the correctness of your algorithm.

4.
   a) In our discussion of the assignment problem, it was observed that subtracting any constant from a row or column does not change the identity of the optimal solution. Does a similar result hold for the transportation problem? Give proof or counterexample.
   b) Using the primal-dual algorithm, find an minimum cost transportation schedule for the following data:

$$\begin{array}{cccc} & 2 & 3 & 3 \\ 3 & 2 & 7 & 1 \\ 1 & 8 & 2 & 8 \\ 4 & 1 & 8 & 2 \end{array}$$

   If you start with $y = 0$ as the dual feasible solution, how many improvements are made to y before optimality is reached?
   c) Assume all the data (cost matrix, supplies, demands) for a transportation problem are positive integers, as in the example above. Show that the primal-dual algorithm must terminate. [Hint: Find a positive constant α such that the dual objective improves by at least α.]