

Local Transformations in Hexahedral Mesh Generation

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Contents

- What is a mesh and why hex meshing ?
- Some known theoretical results.
- Why it is a hard problem ?
- Literature survey.
- Adding more heuristic methods.
- Future plans.

What is a mesh ?

- A mesh is a discretization of space into small simplicial or non-simplicial elements.
 - ***Simplicial Complex***: Triangle and tetrahedra
 - ***Non-simplicial complex***: Quadrilateral, polygon hexahedra, polyhedra.
- Meshes are used in:
 - Solving partial differential equations.
 - Computer animation and simulations.



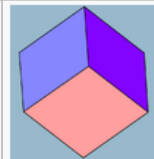
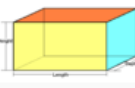
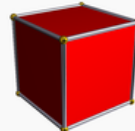
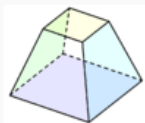
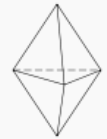
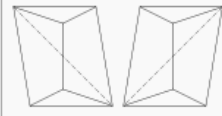
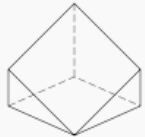
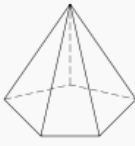
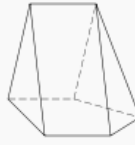

Provable and Automatic Simplicial meshing

- Well-developed theory(especially Delaunay) and robust general purpose software.




Software	Elements	Developer
Triangle	Triangle	Jonathan Shewchuk
Tetgen	Tetrahedra	Hang Si
Quadmesh	Tetrahedra	Tamal Dey
Jaal	Quadrilateral	Tim Tautges

- All these can generate millions of high quality elements/min in any complex geometry.
- A general purpose automatic hex meshing is far legging behind in both theory and practical implementation: widely considered ***the last Holy grail*** in mesh generation.

What is a Hexahedron ? (wikipedia)

Quadrilaterally-faced hexahedra 4^6 faces, 12 edges, 8 vertices					
 Parallelepiped (three pairs of parallelograms)	 Rhombohedron (three pairs of rhombi)	 Trigonal trapezohedron (congruent rhombi)	 Cuboid (three pairs of rectangles)	 Cube (square)	 Quadrilateral frustum (apex-truncated square pyramid)
Others					
 Triangular dipyramid 3^6 Faces 9 E, 5 V	 Tetragonal antiwedge. <i>Chiral</i> – exists in "left- handed" and "right- handed" mirror image forms. 4.4.3.3.3.3 Faces 10 E, 6 V	 4.4.4.4.3.3 Faces 11 E, 7 V	 Pentagonal pyramid 5.3 ⁶ Faces 10 E, 6 V	 5.4.4.3.3.3 Faces 11 E, 7 V	 5.5.4.4.3.3 Faces 12 E, 8 V

There are three further topologically distinct hexahedra that can only be realised as *concave* figures:

Concave		
 4.4.3.3.3.3 Faces 10 E, 6 V	 5.5.3.3.3.3 Faces 11 E, 7 V	 6.6.3.3.3.3 Faces 12 E, 8 V

A hexahedron is any polyhedron with six faces

A Canonical Hexahedron

- A topological cube with distinct 8 vertices, 12 edges, 6 facets.
 - For many FEM methods, all edges should be straight lines, and all faces must be planar convex quadrilaterals.
 - In CAT(k) cubical complex, all edges must be geodesics
- **Numerical methods requirement:**
 - Jacobian must be **positive** at every node i.e. every node must be locally convex.

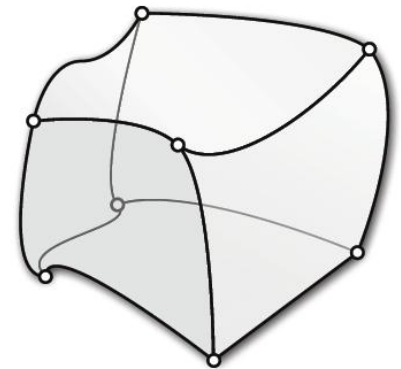
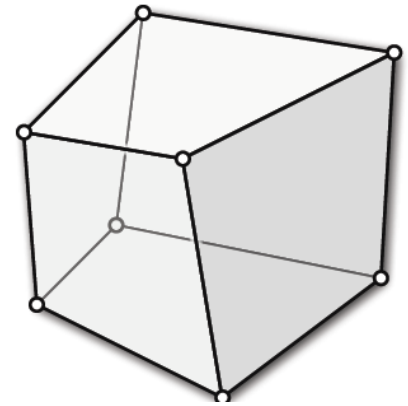


Image: Jeff Erickson

Hex meshes in the past

- In the 70s and 80s.
 - Computer resources were expensive.
 - NASA space program used *Block structured Grids*.
 - Unstructured mesh generation and numerical methods were in an incipient state.
 - Simple geometry, simple mesh, simple numerical schemes.

Cubical Complex in 2012

Puss in Boots, Dreamworks



Animation and Game industries have overtaken FEM community as the biggest users of meshes.
(from Jonathan Shewchuk's book)

Why hex mesh ?

- **Performance:** If the accuracy of numerical simulation is based on #nodes rather than #elements, then a hex mesh is 4-10 times more efficient than a tet mesh.
- **Directional Control:** Hex meshes are superior in direction dominated physical simulations (Boundary layer, shock waves etc).
- **Tensor Product:** Hex mesh exploit tensor product structure.
- **Increase degree of freedom:** Tetrahedral elements have less degree of freedom, therefore, many elastic simulations result in *shear and volume locking*. (looking for video)

What is the hex meshing problem ?

- **Unconstrained problem:**
 - Given a 3D volume, subdivide it into topological and geometric all-hex valid elements.
- **Constrained problem:**
 - Given a quadrilateral boundary mesh, subdivide the interior of the domain with all-hex elements with two conditions:
 - Do not modify the surface mesh.
 - Every quadrilateral element on the surface must be a sub-complex of some hexahedron.

Unconstrained Topological Hex meshing

- Given a 3D volume with triangle or quadrilateral elements:
 - If given quadrilateral mesh, divide each quad into two triangles.
 - Generate constrained tetrahedral mesh which is a solved problem.
 - Convert each tetrahedron into four hexahedra.

**Completely automatic with some
guaranteed (known) quality**

Why constrained hex mesh ?

- Surface quadrilateral mesh encode geometry which is orthogonally another hard problem.
- So that known domain decomposition methods could be applied.
- Different subdomains are independently generated and have different requirements.
- Avoids *domino* effect.
- Locality and reusability.

Where is the problem ?

- Large number of faces have either very high dihedral or very small dihedral angles.
- One excellent paper by Jonathan Shewchuck

What is a good element ? Interpolation, conditioning and Quality measures.

- The numerical errors in gradient calculation can be arbitrary large as dihedral angles increase.
- Conditioning number depends on the small dihedral angles. Poor conditioned matrices affect linear equations solvers by slowing them down or introducing large roundoff errors.

Tet->hex mesh is both inefficient and inaccurate.

Geometric Hex meshing

- **Open problem:** Generate hex mesh with guaranteed practical quality.
- Note that:
 - Perfect hex meshing is impossible by Euler characteristic formula.
 - Even for volume without holes, presence of irregular nodes is must to reduce distortion and density transitions.
 - Topology and geometric qualities are not independent.
 - “Useful” and “practical” quality terms are fuzzy.

Hex meshing

- Since geometric quality measures are volume, Jacobian, edge lengths etc. they can be optimized with non-linear solvers.
- Therefore, a hex meshing problem is essentially a topological problem.

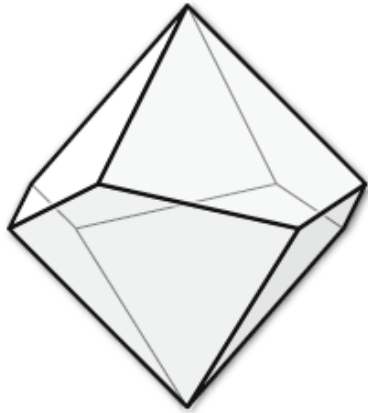
Unbounded High quality.

Will it solve all the problems ?

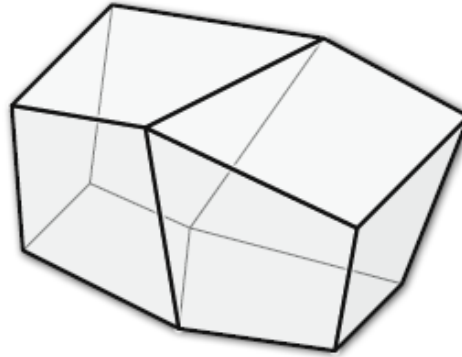
- In general, we can not escape from singular edges in a hex mesh, but can we mitigate the influence of irregularity by increasing the number of extremely high quality elements in the domain ? (Which is doable with boundary pillows, dicing operations).
- We can reduce the percentage of bad elements as low as we want.
- Should singular edges be clustered or evenly placed in the domain.

Answer: I do not know.

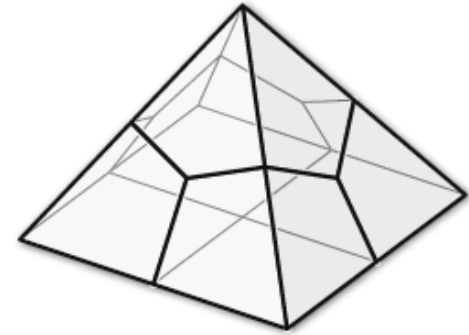
How hard is Hex meshing ?



Octagonal spindle



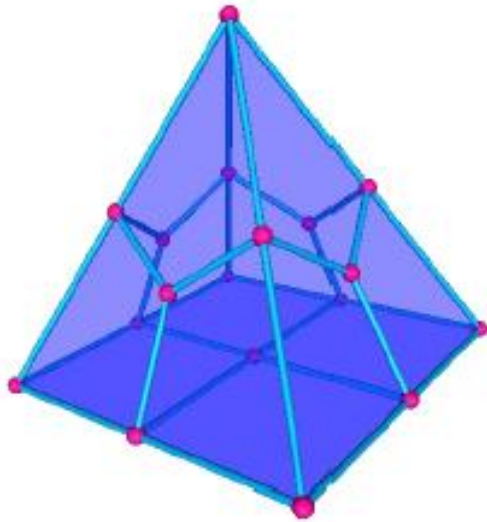
Bicuboid



Schneiders' pyramid

First two are under Open problems category. For Schneider Pyramid, known methods produce warped elements.

Schneider 88 Element Solution



Scheme	#Hex Elements
Yamakawa	88
Suzuki	146
Carbonera(Four Split)	1216
Carbonera(Geode template)	2080

Mitchell-Thurston Existence Theorem

A quad mesh Q of a topological sphere can be extended to a hex mesh of the ball if and only if the number of quads in Q is even.

Elementary Proof:

h : number of hex elements:

f : number of faces

b : number of boundary faces

$$6h = 2f - b$$

Eppstein Complexity

- A topological mesh has complexity of $O(n)$
- A geometric mesh has complexity of $O(n^2)$

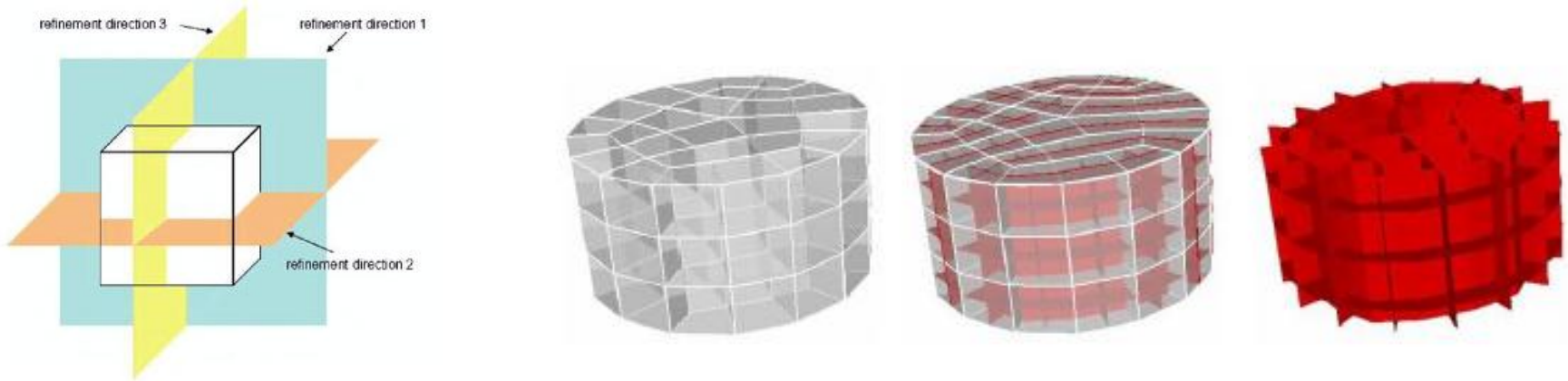
Eppstein writes:

These results are not practical in themselves: the number of elements is too high and we have not satisfactorily completed the solution to the geometric case. Practical hex meshing algorithms are still heuristic and will often fail or require input boundary to be modified.

There is a possibility of a two way interaction between theory and practice: as heuristic methods improve they may solve finite number of geometric cases and thereby prove that all even-quadrilateral polyhedra can be meshed.

Impractical proof of existence can be helpful in guaranteeing that any incremental heuristic method will not get stuck in a bad configuration.

Primal and Dual Complex (images from Jason Shepherd)



- Sheets are 2-manifold surfaces. Each sheet represents a layer of hex elements in the primal mesh.
- Sheets encode global topology of the mesh.
- Complexity of hex meshing is due to sheets structure in the complex.
- A **chord** is regular curve formed by the intersection of one or more dual sheets.
- Fundamental sheets encode the surface geometric features.

Geometric constraints on Dual complex

(Jason Shepherd)

1. Minimize sheet curvature.
2. Maximize orthogonality of sheets.
3. Maximize topologic regularity of sheets.

Some Questions:

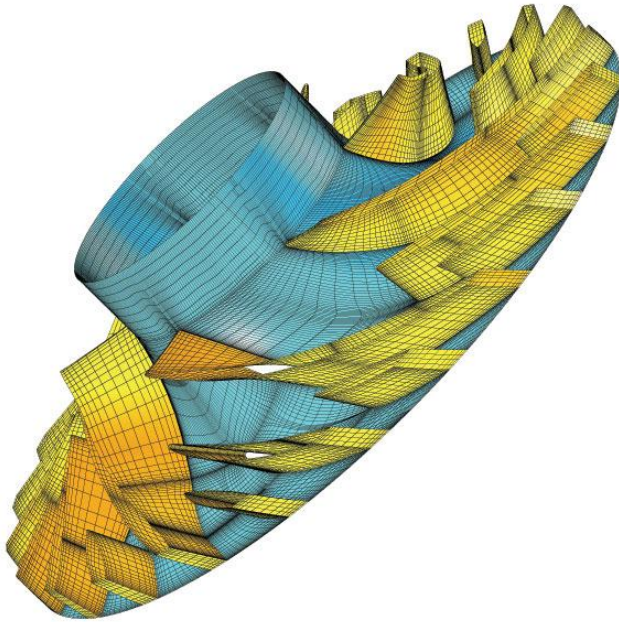
1. Can self-intersecting, self touching, highly tangled sheets have high quality primal mesh ?
2. How surface quadrilateral mesh influence sheet structures ?

Why hex meshing is hard ?

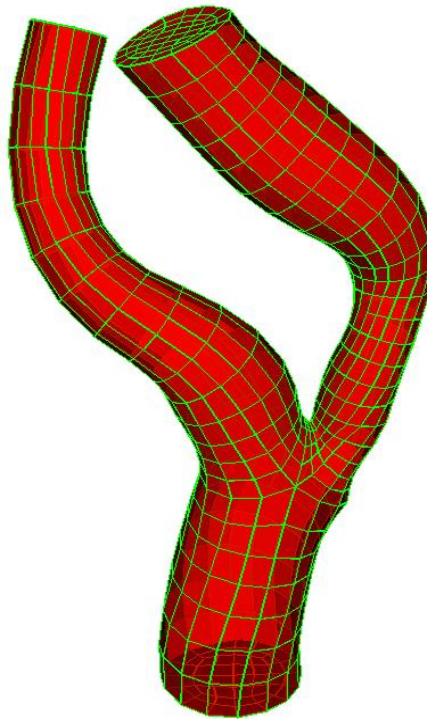
- **Direct methods:** There is no guarantee that some space can be filled with hexahedral elements. Most of the methods start well in the beginning, but in the end land up in a configuration which is hard to fill with hex elements.
- **Indirect methods:** Difficult to modify sheet structures. Some sheet may span the entire geometry (or large percentage) therefore, we can not touch them without changing the geometric model.

Hex meshing

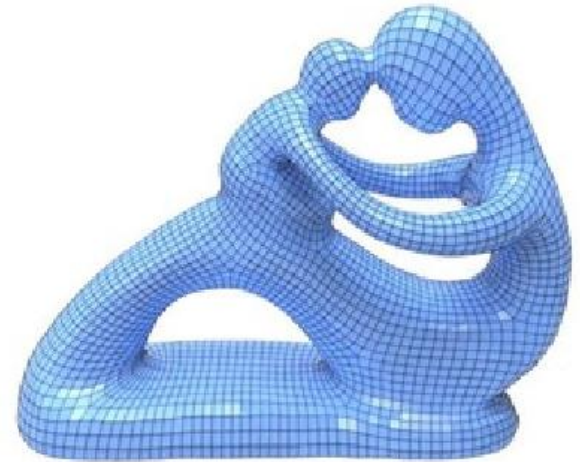
Pointwise



Biomes



Singular Field



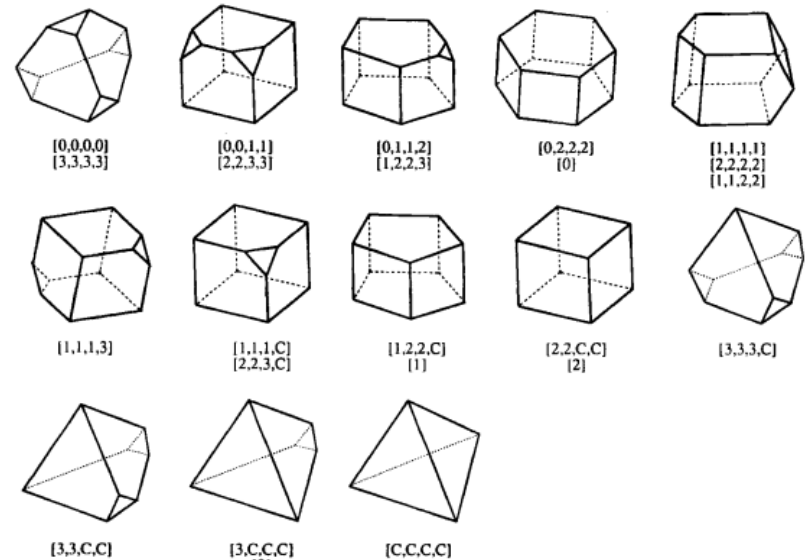
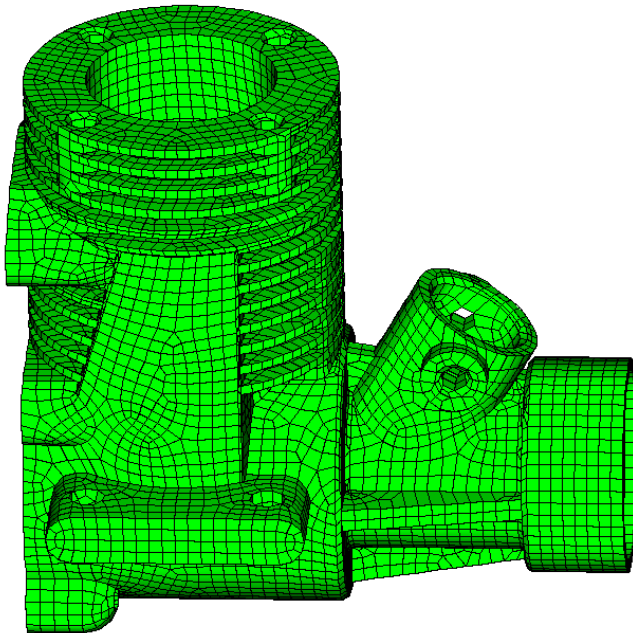
Large number of heuristic methods. No algorithm for automatic meshing.

Heuristic Hex meshing techniques

- Domain decomposition
- Harmonic decomposition.
- Sweeping
- Paving, plastering, unconditional plastering.
- Whisker weaving.
- Octrees, Polycubes
- Mapping
- 3D parameterization
- Subdivision methods

Domain Decomposition

- Break the model into smaller and easily *meshable* components.



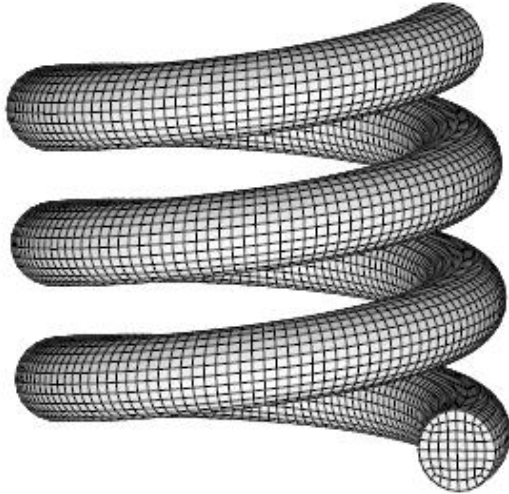
Armstrong 1995, 1997

Harmonic Decomposition



- Specific to tubular and bifurcation models.
- Solve three heat conduction problems to generate mesh in each branch.
- Generates non-intersecting slices, generate hex mesh from the template mesh.

Sweeping



Xevi Roca 2004

- Source and target must be topologically equivalent.
- Can be generalized to multi-source, multi-target using the *Cooper* algorithm.
- In general, produce very high quality hex mesh.

Plastering

Try to extend a highly successful 2D technique (Paving)

- Quadrilateral mesh is generated on the boundary surfaces.
- Advance the fronts in the inward direction.

Problem: Collision of opposite fronts create a configuration which is difficult to fill with hex elements.

Plastering is rarely able to resolve unmeshed voids.

(Staten, Owen and Blacker)

Unconstrained Plastering

- Plastering creates un-meshable spaces, may be because we started with an unsuitable boundary surface mesh.

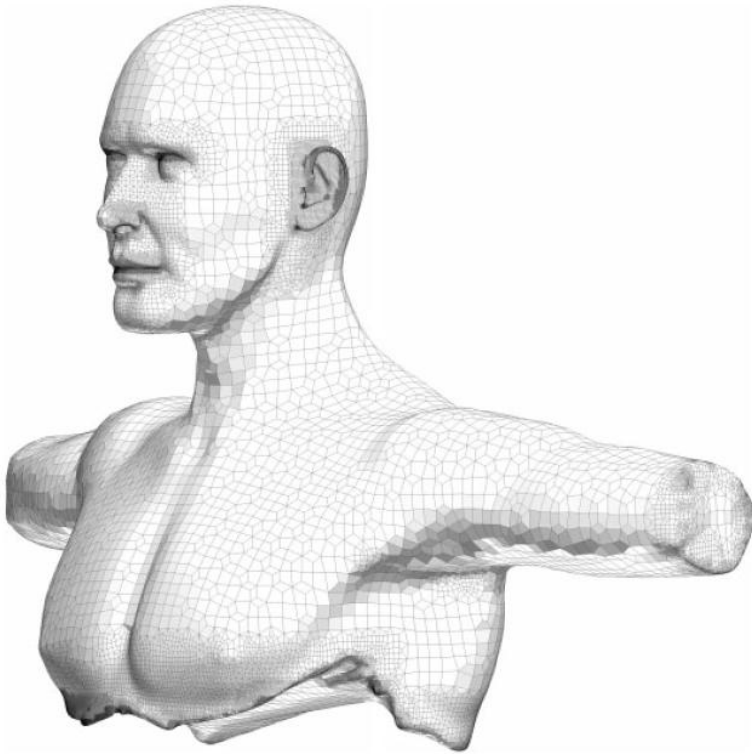
Why do not we allow the algorithm to change the boundary mesh, if it helps avoiding the problematic region in the interior ?

- No known implementation even after many years.
- No proof of reliability.

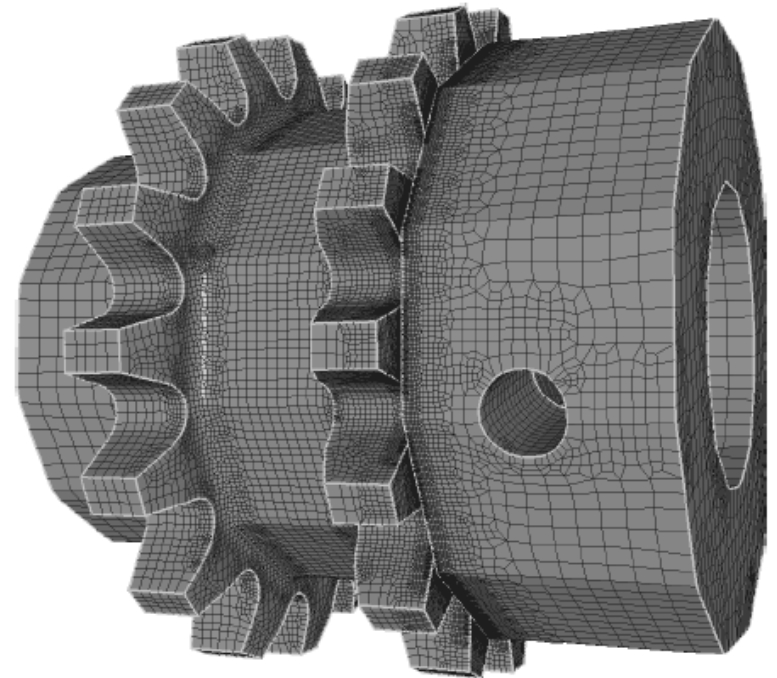
Whisker Weaving

- Plastering and unconstrained methods did not work, Tautges adopted a different approach :
 - Use the surface quadrilateral mesh,
 - Construct hex element connectivity by crossing the dual entities while advancing into the solid.
 - Transform dual complex into primal mesh.
- Often invalid connectivity arises which may be hard to resolve.
- Demonstrated only for simple block-type geometries.

Octrees



Yasushi Ito, Alan Shih and Bharat Soni,
2008



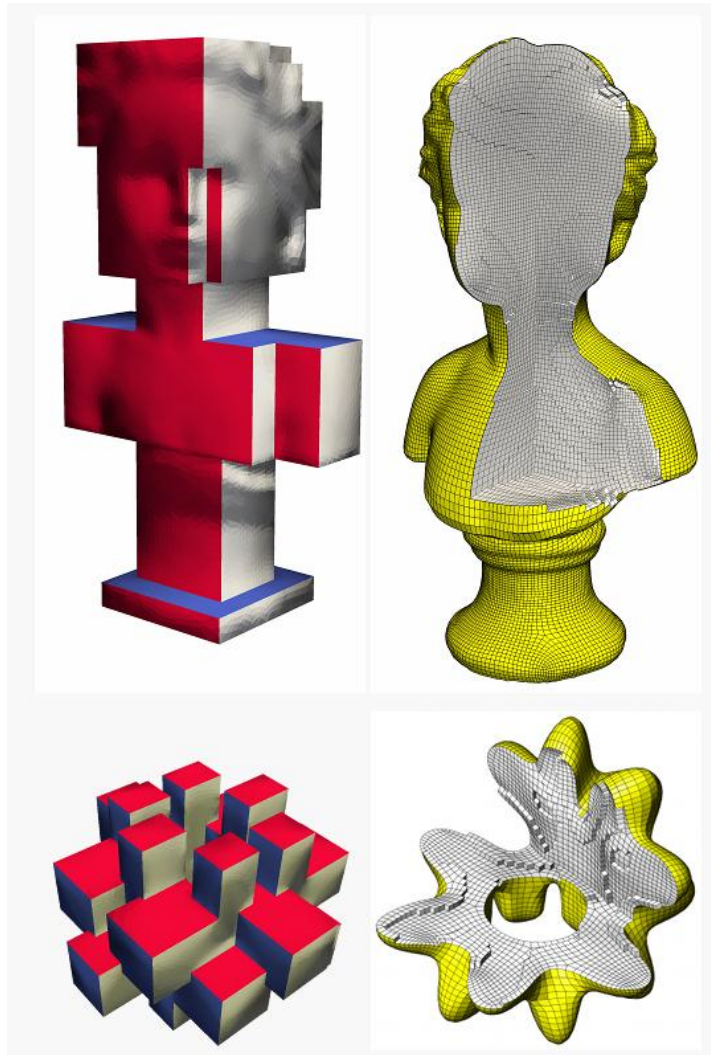
Loic Marechal, 2009

Octree: A new idea in a café

- Traditional octree methods have all the irregular nodes on the boundary surface.
- Scott Mitchell gave a new idea:
 - From a given good surface quadrilateral mesh nodes, generate octree.
 - Now fill the gap between octree and surface with tetrahedral mesh and convert each tet into four hex.
 - Apply geode template and push all the irregular nodes in the interior of the domain.

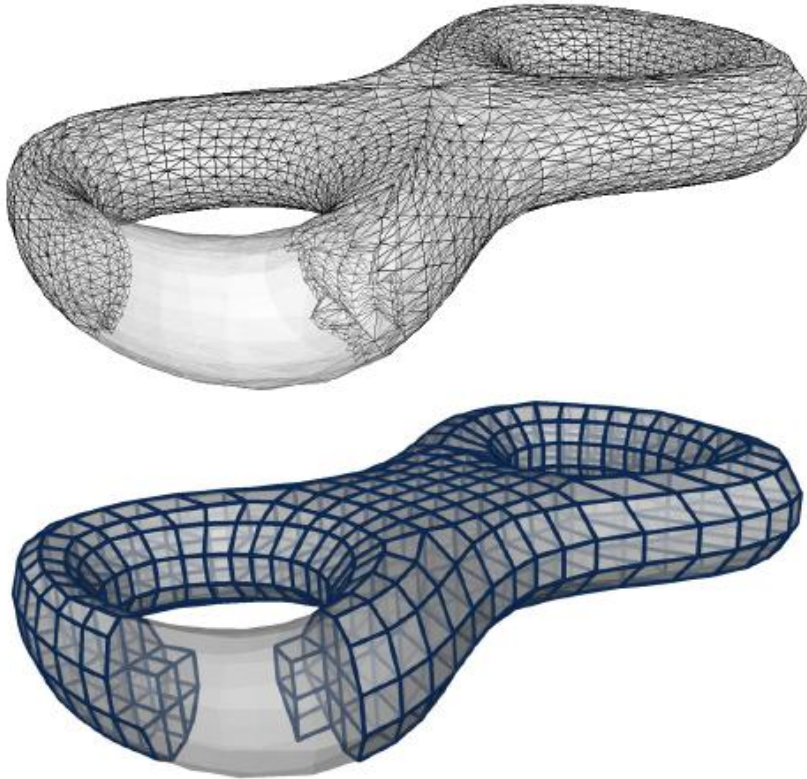
Very nice idea, will implement very soon.

Polycubes



- Model formed by joining large number of equal cubes.
- Element distortion is a big challenge.
- Extending to complex shapes is an open problem.

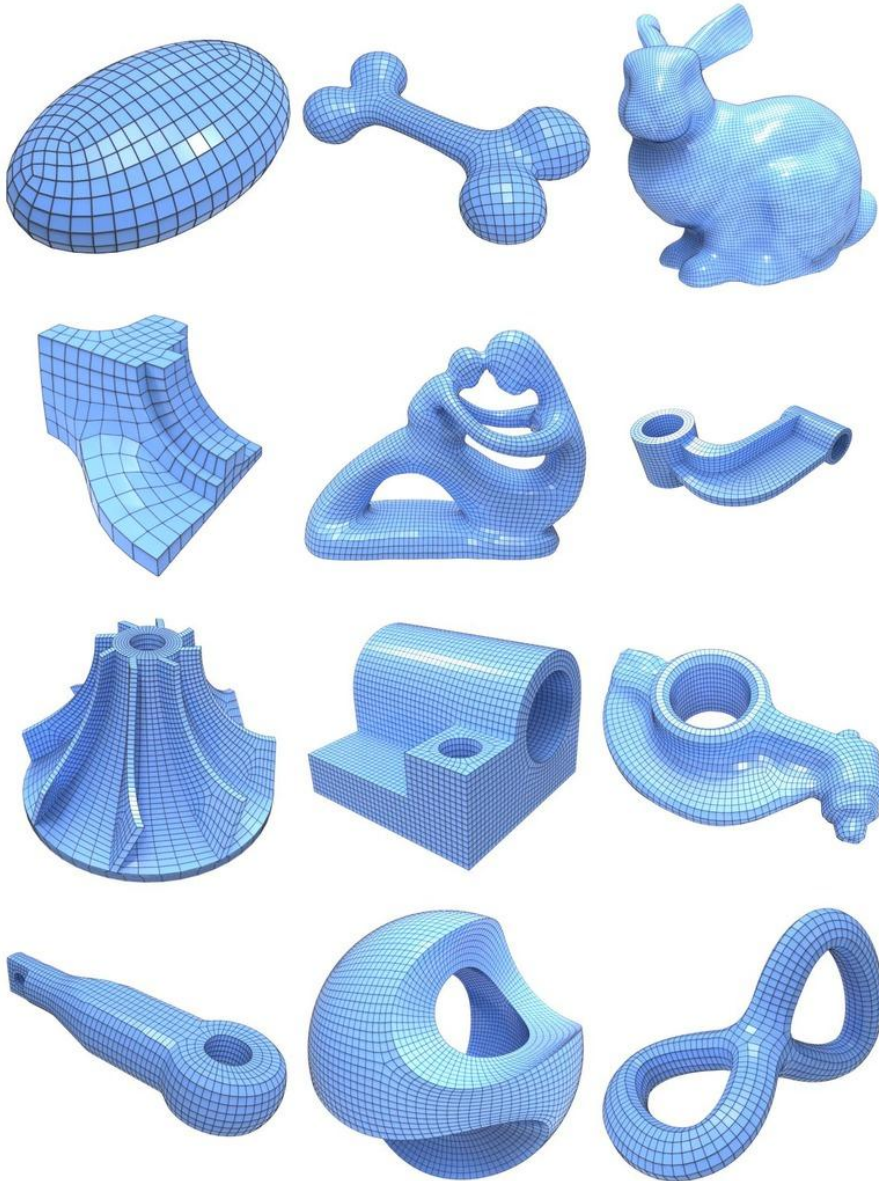
Parameterization



Nieser etc. 2011

1. Input: Tetrahedral mesh.
2. Generate frame field with user defined constraints.
3. Two orthogonal directions of the frame field is defined by curvature of the geometric model and third is normal to the surface.
4. Hex mesh is derived from the field.

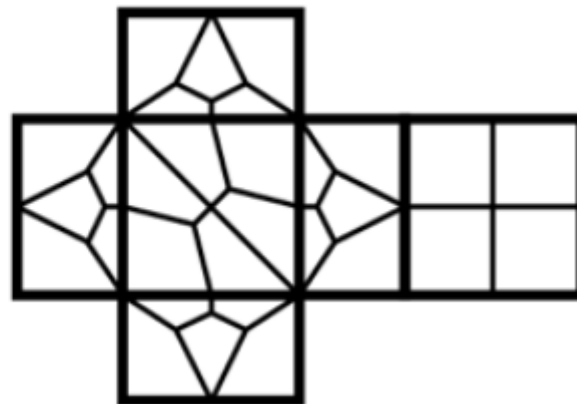
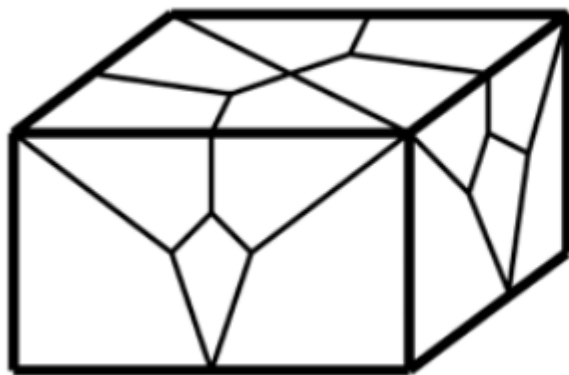
Frame fields



- Best known, non-trivial hex mesh data available
- Linear complexity (constant factor 3-4)
- Passed through all the Mesquite quality criteria.
- High quality sheets (tangle free, non-intersecting etc).

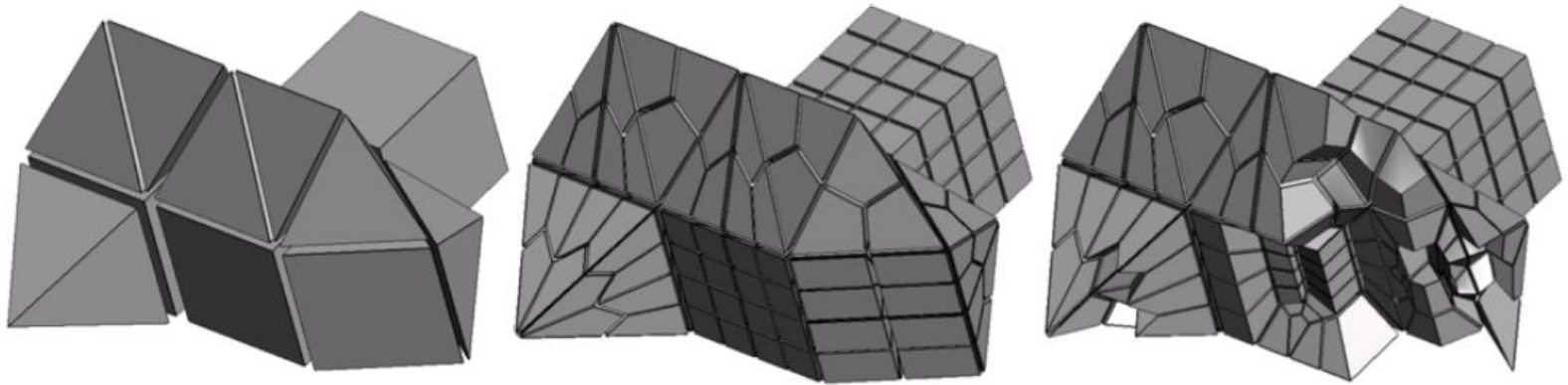
Mitchell's Geode template

- First template to address conforming mesh.
- Never been exploited, but I see good potential now.

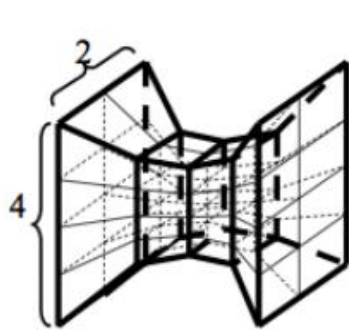


Yamakawa and Shimada Hexhoop templates

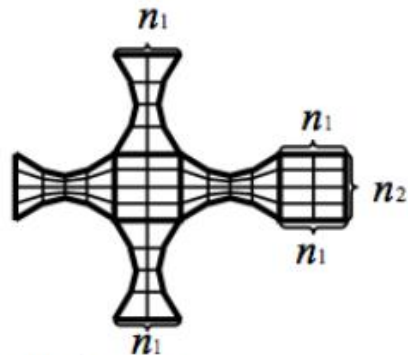
- Set of templates to convert mixed elements into hex elements.
- Just like Geode templates, they are dormant.



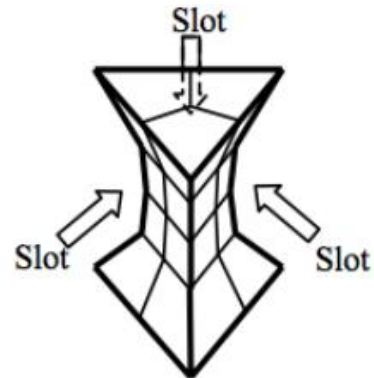
Hexhoops



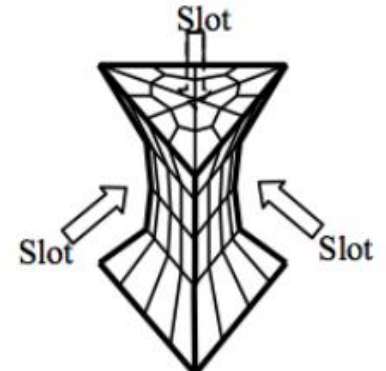
(a) 2x4 rectangular core



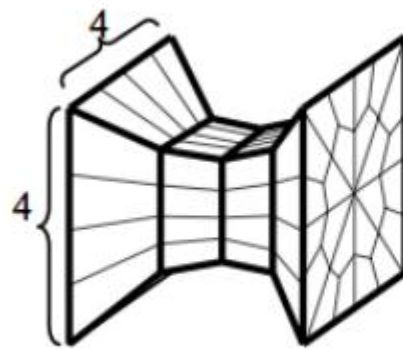
n_1



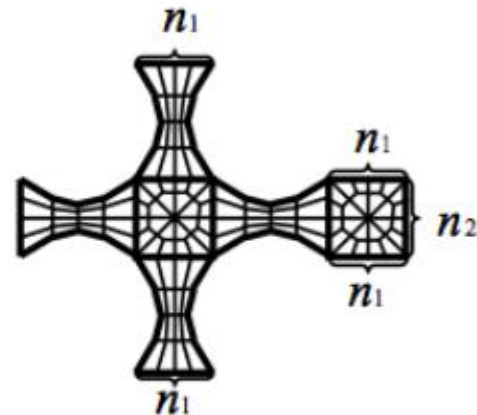
(c) a 2x prism core



(d) A 4x prism core



(b) 4x4 triangular core



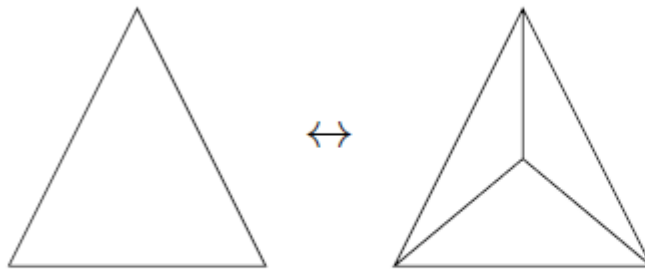
n_1

Disadvantages of Ad-hoc approaches

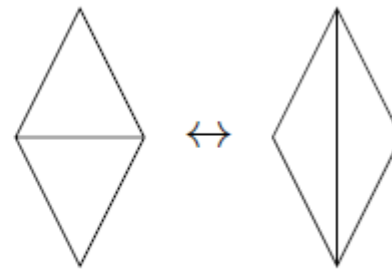
- **Extremely time consuming:**
 - For many complex geometries 30-40 man years efforts not uncommon. (Scott Mitchell, Sandia Lab)
- **Modifications difficult:**
 - Any late stage requirement changes, could mean starting a project from fresh.
- **Experience Driven:**
 - The mesh quality heavily depends on the experience and creativity of the developer.

Pachner moves (or Bi-Stellar flips)

- **Theorem:** Any two triangulations of d -manifolds can be related by finite sequence of *Pachner moves*.



1 ↔ 3



2 ↔ 2

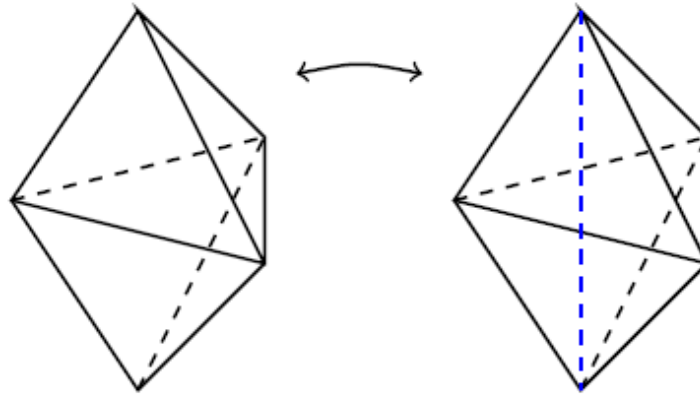
Local Transformations

- Small footprint of atomic operations
 - **Coarseing operations.**
 - **Optimizing operations:** Change local connectivity.
 - **Cleaning operations.** Remove degenerate elements.

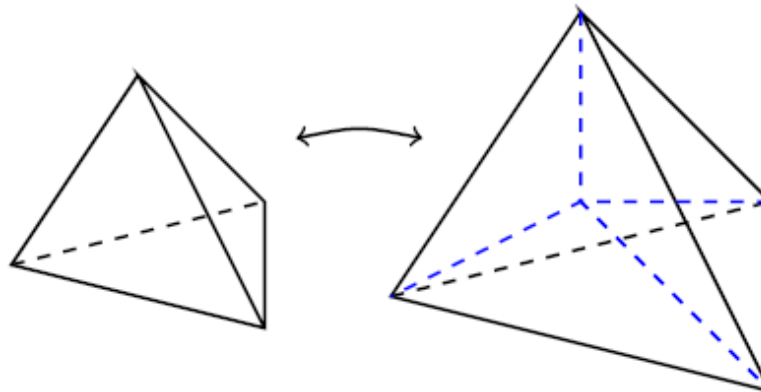
Core of many highly successful in triangle, quad or tetrahedral mesh generation algorithms.

3D Pachner moves

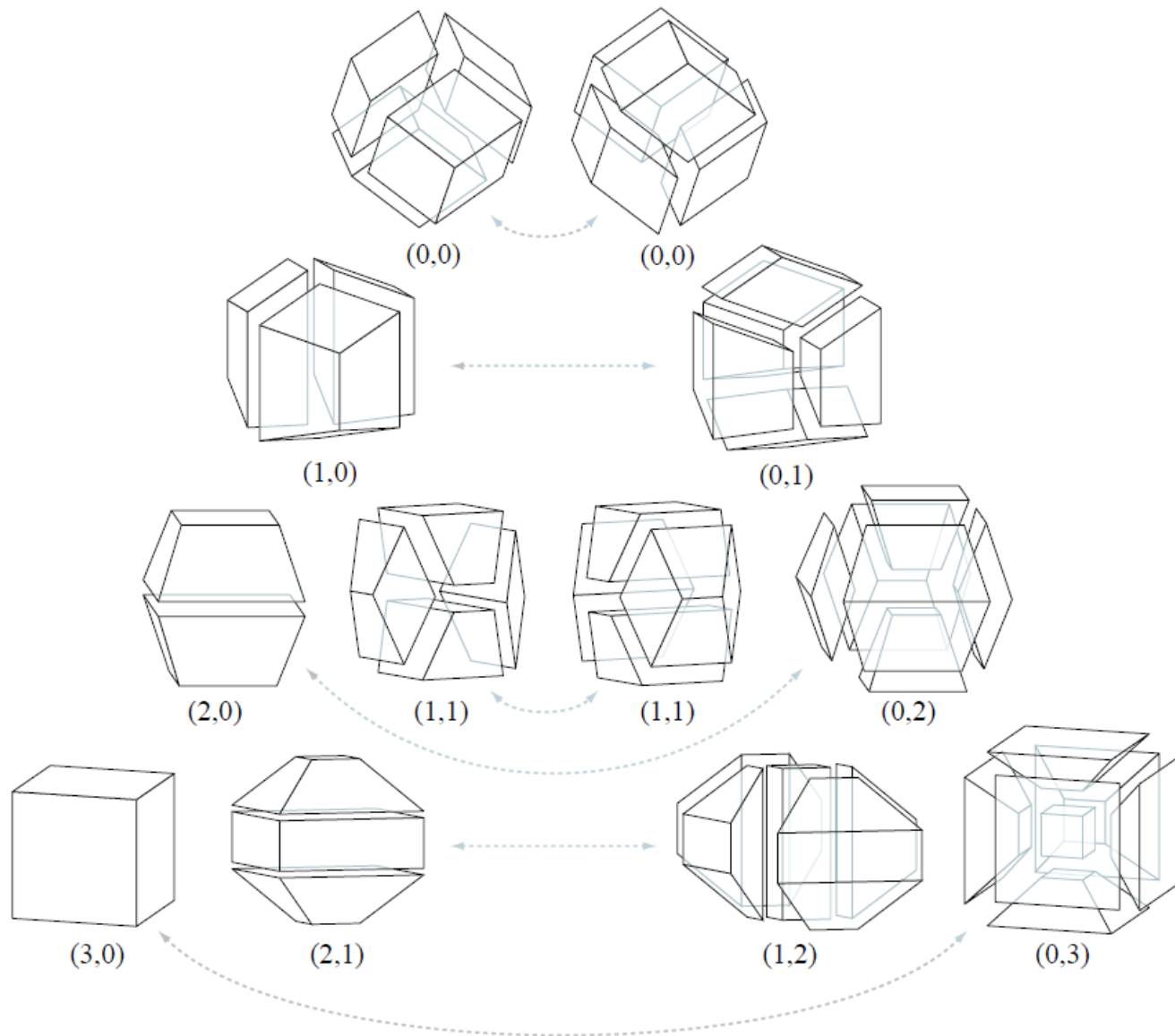
The 2-3 Move:



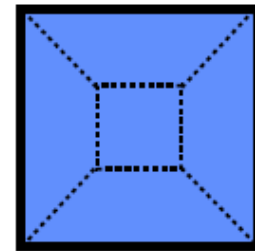
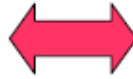
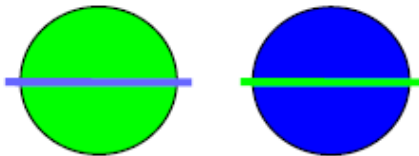
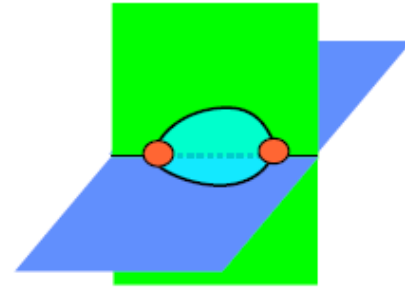
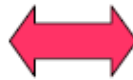
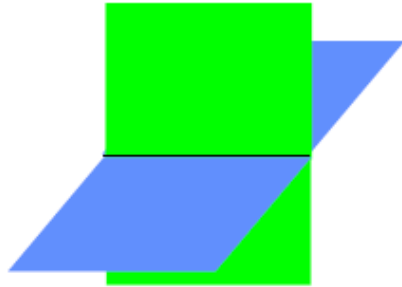
The 1-4 Move:



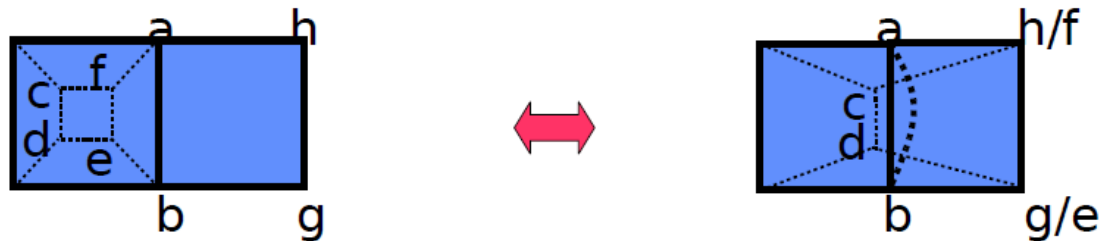
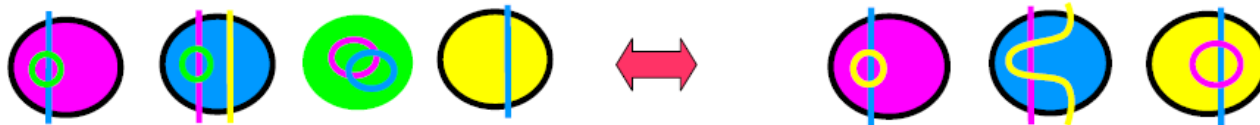
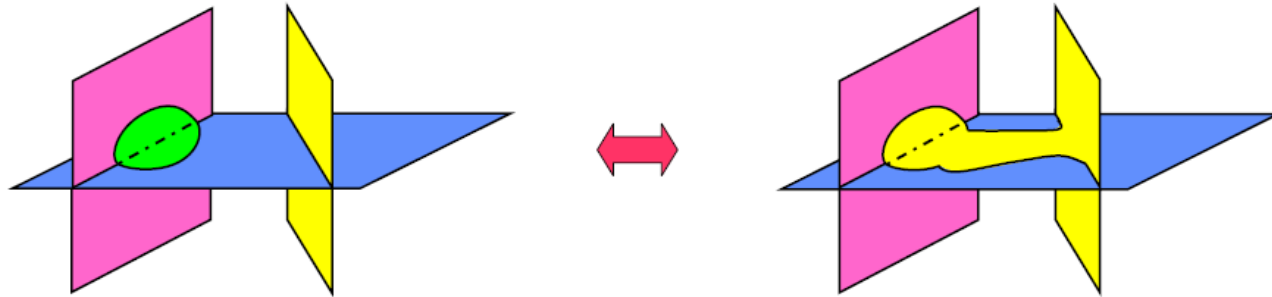
Bern's Operations



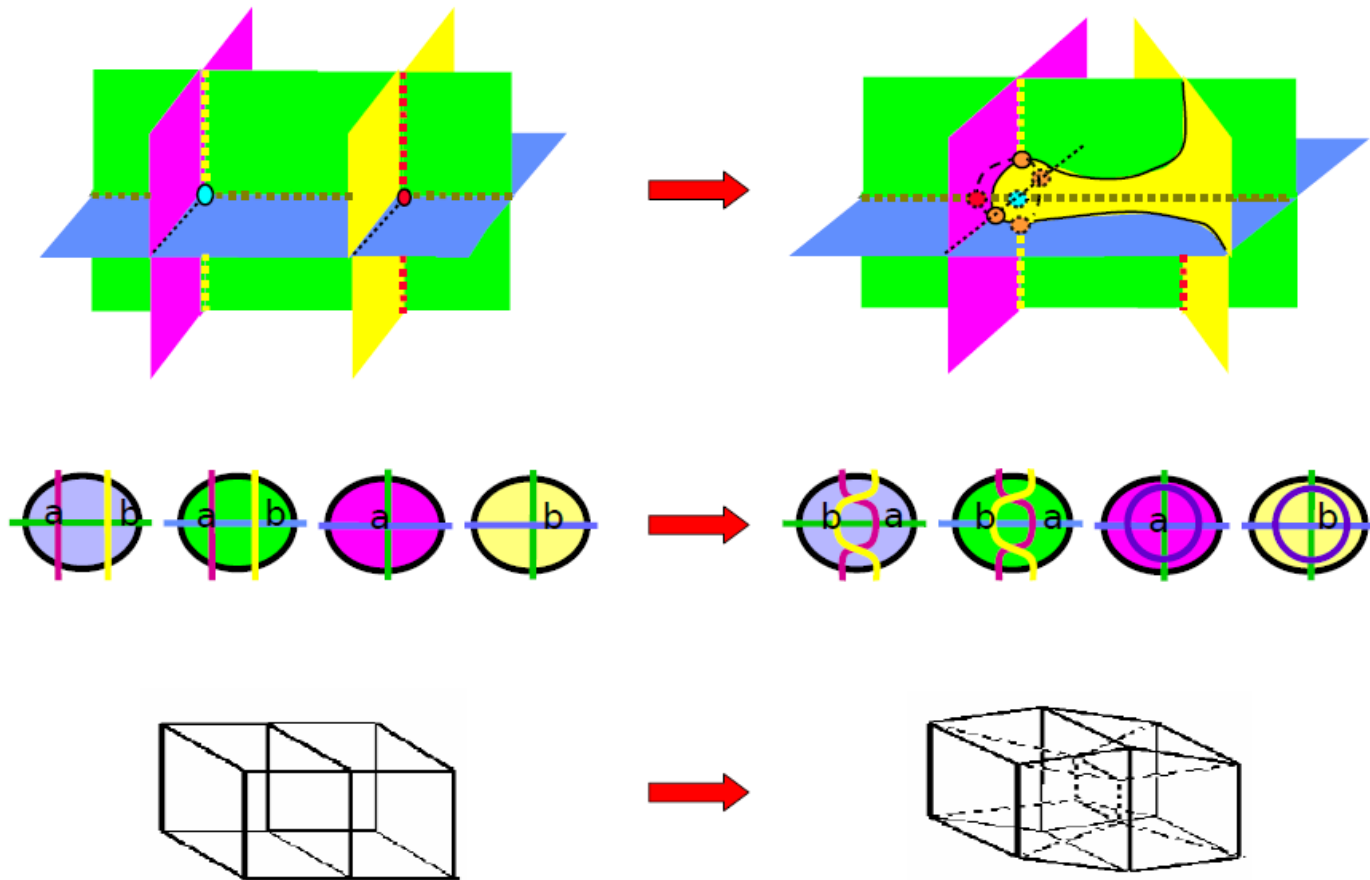
Tautges Atomic operation(1): Atomic Pillow



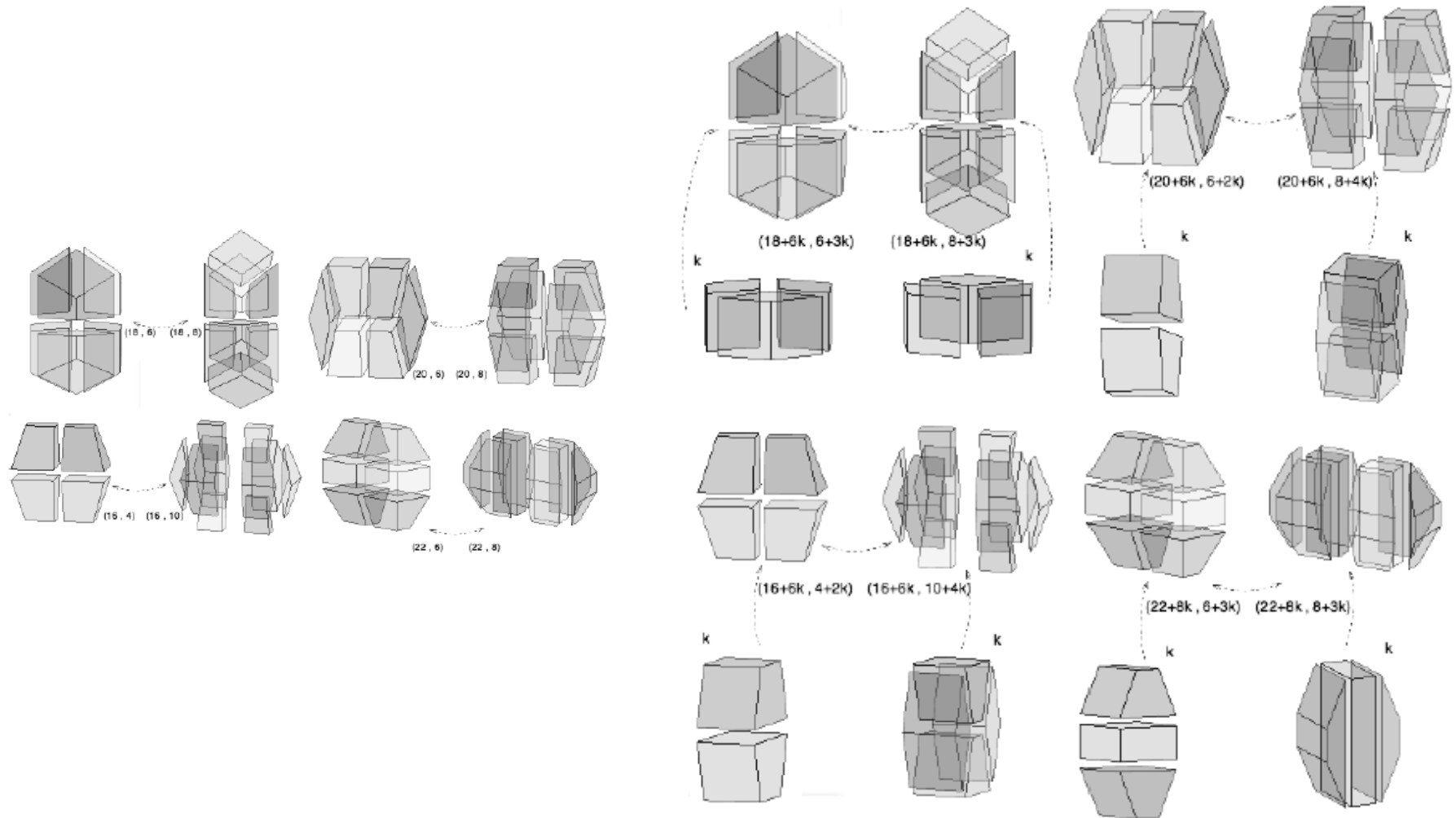
Tautges Atomic Operation(2): Face open Collapse



Tautges Atomic Operation(3): Face Shrink.



Hecht's transformations



Diagonal flips of triangulations on the sphere

(Hideo Komuro 1996)

Any two triangulations with n vertices on the sphere can be transformed into each other, up to ambient isotopy, by at most **$8n-54$** diagonal flips if $n \geq 13$, and by at most **$8n-48$** diagonal flips if $n \geq 7$.

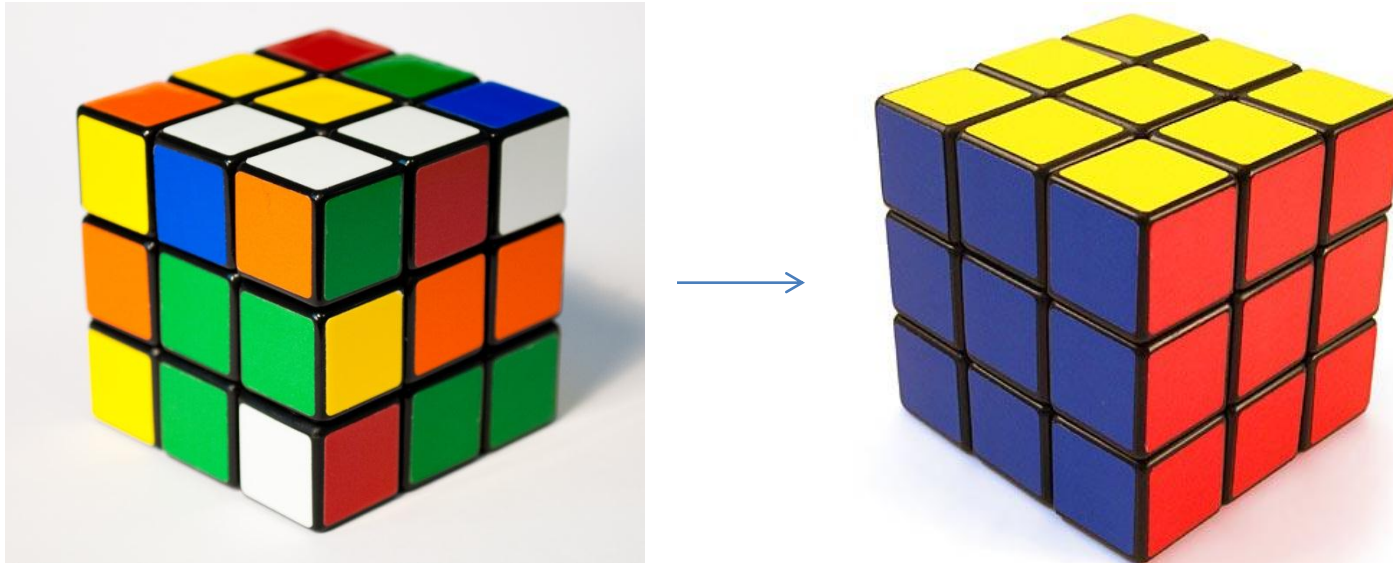
Habegger Question.

Given two PL cubical manifolds, are they related by flipping operations ?

- I am not aware of any theoretical proof for 3D cubes or any experimental results.

How hard can be combinatorial flips ?

God's algorithm for the Rubic Cube.



How many flips are required for 43,252,003,274,489,856,000 positions of the cube ?

After 30 years and large computational resources provided by the google, in year 2010

God number is 20

Complexity of Pachner flips ?

- Let T be a triangulation of a 3-sphere and let t be the number of tetrahedra in it. Then we can simplify the triangulation T to the canonical triangulation of S^3 by making less than $a t^2 2^{bt^2}$ Pachner moves where
 - Constant “a” is bounded above by 6×10^6
 - Constant “b” is smaller than 2×10^4

Are cubical flips simpler than tetrahedral flips ?

God's algorithm v/s Pachner flips

- These results give conflicting hopes. Despite large configuration space, God's algorithm gave a 20 move solution. Pachner result dashes the hopes.
- There is some hidden structure in cubical complexes which could be simplified with some mathematical tool, but I do not know yet.

What mathematical tools they used ?

Group theory

Some questions:

1. Is combinatorial hex meshing simpler than rubic cube ?
2. Can we rely on heuristic methods in hex meshing ?
3. Shall we need mathematical tools such as algebra and group theory for hex meshing ?

Bern, Tautges and Hecht operations.

How effective are they ?

- So far Tautges's operations have been effective in deriving Bern's operations.
- Very small probability of finding the patterns where these operations could be applied.
- No known rules to guide the operations.
- There is not a single paper showing that Bern operations have improved the mesh.
 - When applied in isolation, these operations make mesh even worse.
 - No body has shown how to compose these operations aiming at improving the mesh.
- Many researchers believe that these operations are only theoretical and not practical.

Parity Change:

A theoretical game or engineering necessity ?

- Suppose we are able to change the parity then:

Will Bern's operations solve the hex modification problem ?

- No theoretical work, no strong logical answer.

Parity change: Boy surface ?

- Boy surface is one of the way to change the parity, but is this the only way ?
- No, Shepherd and Carbonera developed a template for a cube to change the parity without using a Boy surface but with large number of singular edges.

Some failed alternate attempts:

How should I modify the sheets and improve the mesh ? Hopeless situation.

- Finding suitable algebra of local modification.
- Knots and Braids theory.
- Matroids theory.
- Graph transformations.
- Algebraic topology.
- Hyperplane arrangements.
- Dehn surgery on knots and hyperplanes.

Is it time to revisit Mitchell's Geode template ?

- Why Geode template did not solve the problem (Circa 1999) ?
 - The quality of internal elements were very poor and no robust geometric optimization software were available (Mesquite).
 - No powerful conforming tetrahedral mesh generators were known (Qualmesh, tetmesh).

Should we add one more heuristic and solve some difficult geometric domains ?

Future of Hex meshing:

- Because of cheap and fast computational resources, marginal performance improvement is less critical.
- Higher order tet elements solve the locking problems. (Weingarten 1994).
- Some new advances in numerical algorithms don't need conforming meshes(Suresh Krishnan, 2012)

Self Evaluation

- After extensive literature survey and own failed attempts, I do not think that I have any magic idea to solve this problem.
- Even simple geometries are hard to solve.
- I could not explore exotic algebraic structures which can relate to hex meshing. Bi-algebra, universal algebra, Lie groups, Knot invariants.
- Should have taken few courses in Knots theory, combinatorial geometry, and group theory.

Automatic Hex meshing: Impossible ?

- Yes, it is hard, but nobody has proven its complexity or impossibility.
- Many researchers in private communication say that (1) Hex mesh is not important now (2) All purpose automatic meshing is unlikely to be solved.
- There are very few high quality research papers both in theory and preliminary “acceptable” results.

Future plan? Practical hex meshing

I will not give a perfect mesh, but will give a reasonably good mesh.

- Explore Geode template and push to its limits.
- Work on sheets modifications, pillows, dicing, knives etc.
- Think more about local transformation, if they can be made useful.
- Develop new intuitive visualization tools to simplify dual->primal conversion.
- Work on non-linear optimization for convexity and planar faces.
- Focus first on low valence edges.
- Work on L_∞ norm tet mesher.