1. **3D.1**
Consider the three circles whose diameters are the sides of a given triangle. Show that the radical center of these circles is the orthocenter of the triangle.

**Proof.**
Draw three circles $C_a$, $C_b$, and $C_c$ on the sides $BC$, $AC$, $AB$ of triangle $ABC$ as diameters. Let $H_a$, $H_b$, and $H_c$ be on $BC$, $AC$, $AB$ and assume that $AH_a$, $BH_b$, and $CH_c$ are perpendicular to $A, B, C$, respectively. It suffices to show that the radical axes of three circles are the same as altitudes of $ABC$. By construction, $C_a$ and $C_b$ meet at $C$ and one other point. Let $R$ be the point that is the intersection of $C_b$ with $BC$. Since $AC$ is a diameter, $\angle ARC = 90^\circ$, thus $AR \perp BC$. Therefore, $R = H_a$ and we find that $C_a$ and $C_b$ meet $C$ and $H_c$. With a similar proof, $C_b$ and $C_c$ meet at $A$ and $H_a$. Also, $C_a$ and $C_c$ meet at $B$ and $H_b$. Hence, $CH_c$ is the radical axis of $C_a$ and $C_b$. Similarly, $AH_a$ is the radical axis of $C_b$ and $C_c$ as well as $BH_b$ is the radical axis of $C_a$ and $C_c$. Therefore, we know that the altitudes of $ABC$ are radical axes of $C_a$, $C_b$, and $C_c$. Since they are equivalent, we finally get the radical center of these circles is the orthocenter of the triangle. This ends the proof. ■

![Figure](image.png)
2. 3D.2

In Figure 3.29, the common chord $PQ$ of two circles bisects line segment $AB$, where $A$ and $B$ lie on the circles as shown. If $X$ and $Y$ are the other points where $AB$ meets the two circles, show that $BX = AY$.

**Proof.**

In the figure, the common chord $PQ$ of two circles bisects line segment $AB$, where $A$ and $B$ lie on the circles as shown. Assume that $X$ and $Y$ are the other points where $AB$ meets the two circles. Let $M$ be the point where $PQ$ and $AB$ meet. Then, we have $AM = BM$ by assumptions. By Theorem 1.35, $AM.XM = PM.MQ$ and $PM.MQ = BM.MY$. Since $AM.XM = PM.MQ = BM.MY$, $AM.XM = BM.MY$, thus we deduce $XM = MY$. Since $AM = AY + MY = BM = BX + XM$, $AY + MY = BX + XM$ and we finally get $AY = BX$ because $XM = MY$. This completes the proof. ■
Given three concurrent Cevians in a triangle, show that the three lines obtained by joining the midpoints of the Cevians to the midpoints of the corresponding sides are concurrent.

**Proof.**

Given $\triangle ABC$, let $M_a$, $M_b$, and $M_c$ be midpoints of $BC$, $AC$, $AB$, respectively. Then $M_aM_bM_c$ is the medial triangle. Let an arbitrary point $X_a$ be on $BC$. Draw $AX_a$ from vertex $A$ and let $N_a$ be a point on $M_cM_b$ meeting $AX_a$. Let $N_b$ be a point on $M_aM_c$ meeting $BX_b$ and let $N_c$ be a point on $M_aM_b$ meeting $CX_c$. First, we show $N_a$ is the midpoint of $AX_a$. It suffices to show $M_cN_a \parallel BX_a$ because $M_c$ is the midpoint of $AB$ and $M_b$ is the midpoint of $AC$. By Corollary 1.31, we know $M_cM_b \parallel BC$ because $M_c$ and $M_b$ are midpoints of $AB$ and $AC$, respectively. Since $N_a$ is on $M_cM_b$, $N_a$ is the midpoint of $AX_a$, as required. With similar proofs, $N_b$ is the midpoint of $BX_b$ and $N_c$ is the midpoint of $CX_c$. Therefore, we get $\triangle AM_cN_a \sim \triangle ABX_a$ by AA, $\triangle BM_cN_b \sim \triangle BAX_b$, and $\triangle CM_bN_c \sim \triangle CAX_c$ by AA.

Since these points are midpoints, $M_cN_a = \frac{1}{2}BX_a$ and $N_aM_b = \frac{1}{2}X_aC$ by Corollary 1.31. Thus, we have

$$\frac{M_cN_a}{N_aM_b} = \frac{BX_a}{X_aC}$$

Similarly,

$$\frac{M_bN_c}{N_cN_a} = \frac{AX_c}{X_cB}$$

and

$$\frac{M_aM_b}{N_bN_c} = \frac{CX_b}{X_bA}$$

Since

$$\frac{BX_a}{X_aC} \cdot \frac{AX_c}{X_cB} \cdot \frac{CX_b}{X_bA} = 1$$

by Ceva’s Theorem, we get

$$\frac{M_cN_a}{N_aM_b} \cdot \frac{M_bN_c}{N_cN_a} \cdot \frac{M_aM_b}{N_bN_c} = 1$$

Since the products are 1, by Ceva’s Theorem, we find that three lines obtained by joining the midpoints of the Cevians to the midpoints of the corresponding sides are concurrent, as required. ■