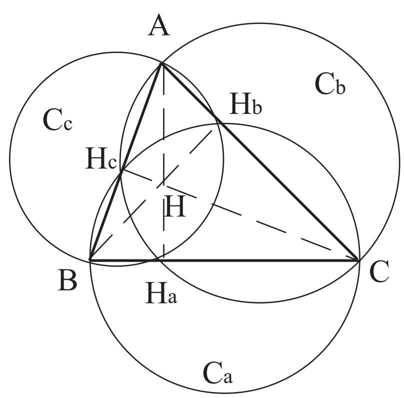
1. ***3D.1***
Consider the three circles whose diameters are the sides of a given triangle. Show that the radical center of these circles is the orthocenter of the triangle.

Proof.
Draw three circles C_a, C_b, and C_c on the sides BC, AC, AB of triangle ABC as diameters. Let H_a, H_b, and H_c be on BC, AC, AB and assume that AH_a, BH_b, and CH_c are perpendicular to A,B, and C, respectively. It suffices to show that the radical axes of three circles are the same as altitudes of ABC. By construction, C_a and C_b meet at C and one other point. Let R be the point that is the intersection of C_b with BC. Since AC is a diameter, $\angle ARC = 90°$, thus $AR \perp BC$. Therefore, $R = H_a$ and we find that C_a and C_b meet C and H_c. With a similar proof, C_b and C_c meet at A and H_a. Also, C_a and C_c meet at B and H_b. Hence, CH_c is the radical axis of C_a and C_b. Similarly, AH_a is the radical axis of C_b and C_c as well as BH_b is the radical axis of C_a and C_c. Therefore, we know that the altitudes of ABC are radical axes of C_a, C_b, and C_c. Since they are equivalent, we finally get the radical center of these circles is the orthocenter of the triangle. This ends the proof. ■



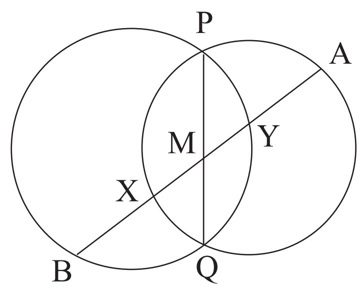
Figure

2. ***3D.2***

In Figure 3.29, the common chord PQ of two circles bisects line segment AB, where A and B lie on the circles as shown. If X and Y are the other points where AB meets the two circles, show that $BX = AY$.

Proof.

In the figure, the common chord PQ of two circles bisects line segment AB, where A and B line on the circles as shown. Assume that X and Y are the other points where AB meets the two circles. Let M be the point where PQ and AB meet. Then, we have $AM = BM$ by assumptions. By Theorem 1.35, $AM.XM = PM.MQ$ and $PM.MQ = BM.MY$. Since $AM.XM = PM.MQ = BM.MY$, $AM.XM = BM.MY$, thus we deduce $XM = MY$. Since $AM = AY + MY = BM = BX + XM$, $AY + MY = BX + XM$ and we finally get $AY = BX$ because $XM = MY$. This completes the proof. ■



Figure

3. ***4A.5***
Given three concurrent Cevians in a triangle, show that the three lines obtained by joining the midpoints of the Cevians to the midpoints of the corresponding sides are concurrent.

Proof.
Given $\triangle ABC$, let M_a, M_b, and M_c be midpoints of BC, AC, AB, respectively. Then $M_aM_bM_c$ is the medial triangle. Let an arbitrary point X_a be on BC. Draw AX_a from vertex A and let N_a be a point on M_cM_b meeting AX_a. Let N_b be a point on M_aM_c meeting BX_b and let N_c be a point on M_aM_b meeting CX_c. First, we show N_a is the midpoint of AX_a. It suffices to show $M_cN_a \| BX_a$ because M_c is the midpoint of AB and M_b is the midpoint of AC. By Corollary 1.31, we know $M_cM_b \| BC$ because M_c and M_b are midpoints of AB and AC, respectively. Since N_a is on M_cM_b, N_a is the midpoint of AX_a, as required. With similar proofs, N_b is the midpoint of BX_b and N_c is the midpoint of CX_C. Therefore, we get $\triangle AM_cN_a \sim \triangle ABX_a$ by AA, $\triangle BM_cN_b \sim \triangle BAX_b$, and $\triangle CM_bN_c \sim \triangle CAX_c$ by AA.
Since these points are midpoints, $M_cN_a = \frac{1}{2}BX_a$ and $N_aM_b = \frac{1}{2}X_aC$ by Corollary 1.31. Thus, we have

$$\frac{M_cN_a}{N_aM_b} = \frac{BX_a}{X_aC}$$

Similarly,

$$\frac{M_bN_c}{N_cN_a} = \frac{AX_c}{X_cB}$$

and

$$\frac{M_aM_b}{N_bM_c} = \frac{CX_b}{X_bA}$$

Since

$$\frac{BX_a}{X_aC} \cdot \frac{AX_c}{X_cB} \cdot \frac{CX_b}{X_bA} = 1$$

by Ceva's Theorem , we get

$$\frac{M_cN_a}{N_aM_b} \cdot \frac{M_bN_c}{N_cN_a} \cdot \frac{M_aM_b}{N_bM_c} = 1$$

Since the products are 1, by Ceva's Theorem, we find that three lines obtained by joining the midpoints of the Cevians to the midpoints of the corresponding sides are concurrent, as required. ■