CS412 **Assignment 7** Fall 2000

due date: 12 December 2000

STAPLE your homework. MARK your homework clearly with your NAME. In addition, write the first letter of your LAST NAME boldly into the upper left corner of the first page of your homework . For this to be of real help, the vertical edge of the upper left corner should be longer than the horizontal edge.

Your script, along with the relevant m-file you create, is to be handed in also electronically. Submit these two files electronically by copying them into the directory `/p/course/cs412-deboor/homework` (on `afs`). Be aware that you cannot overwrite nor read a file already there, not even your own.

This assignment has to do with the system of ordinary differential equations describing the *restricted* three-body problem. This system describes the motion of a small body affected by the gravitational field of two large bodies that are rotating around each other. An example of this is a space station affected by the earth and the moon.

The coordinate system to be used is a bit tricky. The two large bodies and their motion around each other determine a plane in space, and a two-dimensional cartesian coordinate system is used in this plane. The origin of the coordinate system is at the center of mass of the two heavy bodies, the x-axis is the line through these two bodies, and the distance between their centers is the unit of length. In particular, the coordinate system rotates with the system. Thus, if μ is the ratio of the mass of the moon to that of the total mass of the moon and the earth, then, in this coordinate system, the center of the moon and of the earth are located at coordinates $(1-\mu, 0)$ and $(-\mu, 0)$, respectively.

The third body, the space station, is assumed to have a mass that is negligible compared to the mass of the others. The motion of this third body is restricted to the plane determined by the two larger bodies. The position of the third body at time t is $(x(t), y(t))$. The equations of motion are based on Newton's laws of gravitation and motion. They are

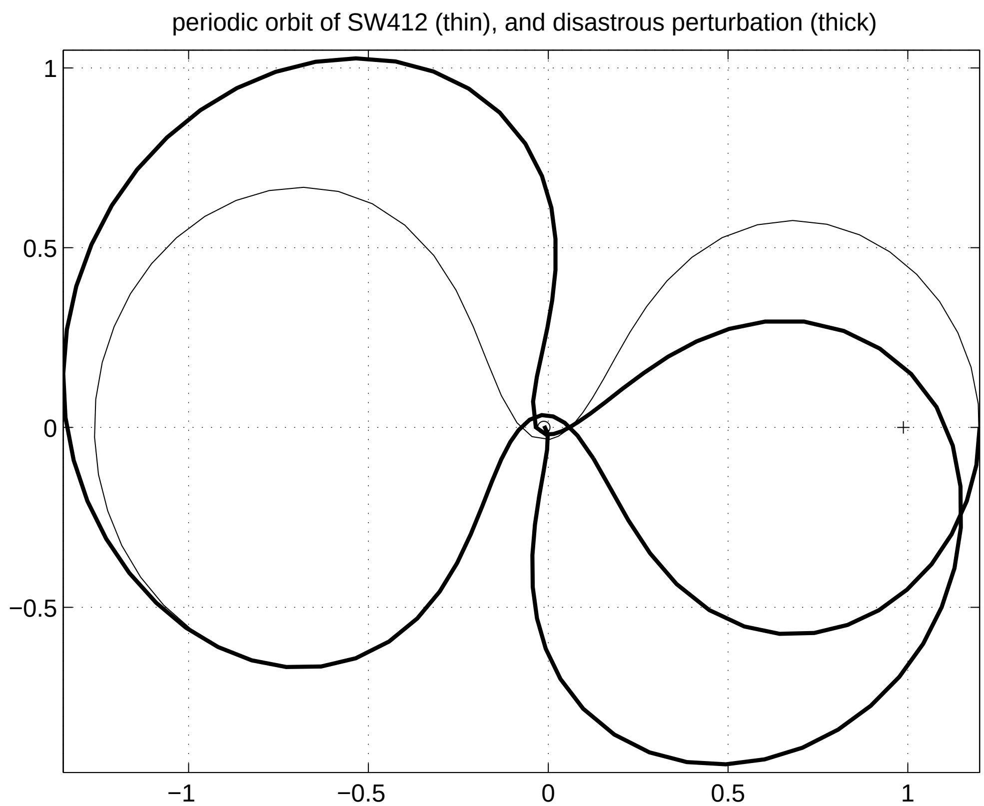
$$\begin{aligned}
x'' &= \quad 2y' + x - \frac{\mu^*(x+\mu)}{r_1^3} - \frac{\mu(x-\mu^*)}{r_2^3} \\
y'' &= -2x' + y - \frac{\mu^* y}{r_1^3} - \frac{\mu y}{r_2^3} \\
\mu &= \frac{1}{84.45}, \qquad \mu^* = 1-\mu \\
r_1 &= ((x+\mu)^2 + y^2)^{1/2} \\
r_2 &= ((x-\mu^*)^2 + y^2)^{1/2}.
\end{aligned} \tag{1}$$

The prime denotes differentiation with respect to time, t. The quantity r_1 is the distance of the space station from the center of the earth, and r_2 is its distance from the center of the moon.

Although a great deal is known about this system of equations, it is not possible to find a nontrivial closed-form solution. One interesting class of solutions are the periodic orbits. These are solutions of the system of ordinary differential equations that repeat themselves after a certain time. One periodic orbit is given by the initial conditions

$$\begin{array}{ll} x(0) = 1.2, & x'(0) = 0.0, \\ y(0) = 0.0, & y'(0) = -1.04625751. \end{array} \tag{2}$$

This orbit has a period T equal to about 6.2. (The unit of time is 1 lunar month, i.e., about 28.15 (earth) days.) This particular orbit starts on the far side of the moon with an altitude of about 0.2 times the earth-moon distance and a certain initial velocity. The orbit leaves the neighborhood of the moon and comes fairly close to earth and then goes beyond it. The gravitational pull of the earth and moon then pulls it back toward the earth; it passes close to the earth, then back to its original position near the moon.



Orbits such as this are potentially useful as orbits of 'space warehouses' which would travel near both earth and the moon, and allow for cheap transportation between the two. Short-range shuttles could be used to transfer goods between the warehouse and the moon or earth.

For the purpose of this assignment, assume you are a space cadet assigned to the orbiting space warehouse SW412. This job is usually pretty boring, but there is trouble ahead! A mysterious force (Klingons??) will affect SW412 on its journey beyond earth. If nothing is done, the effect will be to pull SW412 from its path, having it ultimately crash into earth.

Fortunately, you are in charge of the emergency propulsion system on SW412, hence have the power to prevent this catastrophe. Unfortunately, when you turn on that propulsion system as soon as you are out of that mysterious force field, you discover that (perhaps due to budget cutbacks) the only real control you have is to choose the shut-off time. So, your only hope is to so choose this time of shut-off that, in the next 20 time units, SW412 neither crashes into the earth nor goes off into outer space, but, rather, passes close to earth at least 3 times, giving the controllers on earth a chance to send up a rescue mission.

With the effect of the mysterious force field and the SW412 propulsion system figured in, the differential equations are

$$\begin{aligned} x'' =& \quad 2y' + x - \frac{\mu^*(x+\mu)}{r_1^3} - \frac{\mu(x-\mu^*)}{r_2^3} \\ & + b(t)(x'-y)/((x'-y)^2+(y'+x)^2)^{1/2} - F(t) \\ y'' =& -2x' + y - \frac{\mu^* y}{r_1^3} - \frac{\mu y}{r_2^3} \\ & + b(t)(y'+x)/((x'-y)^2+(y'+x)^2)^{1/2} \end{aligned} \tag{3}$$

The function F describes the simple force field (of mysterious origin); it is given by

$$F(t) = \begin{cases} (t-2)(3-t) & \text{for } 2 < t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

The function b represents your propulsion of SW412. As already mentioned, it is not terrribly effective. It is given by

$$b(t) = \begin{cases} 1 & \text{for } 3 \le t < t^* \\ 0 & \text{otherwise} \end{cases}$$

with the time t^* the only thing under your control. Fortunately, you have taken CS412!

1. (40 points) Use the `matlab` command `ode45` to compute the solution of the system (1) with the initial conditions (2). For this, you will have to convert (1) into an equivalent first-order system, $z'(t) = f(t, z)$, then encode this function $f(t, z)$ in an m-file, `f`⟨*yourlogin*⟩`.m` say, for use in `ode45`. Note that the default error settings in `ode45` will not give you good enough answers, but that a `RelTol` of `1e-6` (set by using the additional argument `odeset('RelTol',1e-6)`) will do.

Compute for time up to 7. Verify that the solution is apparently periodic with period about 6.2. Use cubic Hermite interpolation (a handy m-file, called `pwch.m`, is available in the oct05 diary) to suitably refine the output supplied by `ode45`, then plot the path taken by SW412. (Note that everything is parametrized by time, but, near the earth, SW412 will move much faster than away from earth, hence equal spacing in time is not at all equal spacing of points along the path. In particular, you need to use a finer spacing in time when SW412 is close to earth than when it is away from earth.)

2. (10 points) Show that if nothing is done to change the speed of SW412, then the force field of mysterious origin will cause SW412 to crash into the earth, around $t = 8.085$. For this, use system (3) with t^* some number less than 3, and with the same initial conditions (2). Compute up to time 20.

3. (10 points) Find a value or values of t^* that will have SW412 come close to the earth at least 3 times during the first 20 time units, but without crashing into the earth. Use the system (3) with initial conditions (2), and various choices for t^*. Compute up to time 20 at most. (As a check, for $t^* \le 3.06$, SW412 will crash into the earth.)

> Important! Don't crash your craft into the earth! Assume that the earth is a sphere of radius 3950 miles and that the distance between the centers of the earth and the moon is 239,000 miles. Note that the origin of the coordinate system is within the earth, but is not at its center. To enforce this imperative, put a check into your `f`⟨*yourlogin*⟩`.m` file for this condition and make it return a vector of `NaN`'s in case the current position of the craft is inside the earth. This will cause `ode45` to generate `NaN`'s from then on, and `plot` will ignore these. (A more professional way to deal with this is to use the `events` option in `ode45`, but this might be too much for you to handle.)

what is to be handed in:

Your conclusions should be stated in complete sentences in a short report. The script to be handed in (also electronically) should be set to compute (with the aid of your `f`⟨*yourlogin*⟩`.m` answers to all three problems stated and plot a relevant picture for each of them. Usa the matlab command `figure` to be certain that three separate figures will show on the screen.

The graders will run it by typing the command ⟨*yourlogin*⟩`7` which means that the name of your script file better be your login-name, followed by `7`, followed by the suffix `.m`, and that you will also have handed in (and correctly referred to) the function m-file `f`⟨*yourlogin*⟩`.m` that describes the specific systems.

Note that you get to hand in just **ONE** file `f`⟨*yourlogin*⟩`.m` (along with your script). Since `ode45` expects to call this function with a statement like `yprime = f(t,y)`, this means that you have to use `global` variables to account for the various changes needed (do a `help global`). Specifically, the mysterious force will have to be turned on and off in your script via a global variable, and the cut-off time $t*$ must be supplied as a global variable. (To be sure, `ode45` also offers a way to use an `f` with additional input parameters, and you are free to use that instead of global variables; see `help ode45`.)

4. (3*5 points) For each of the following, state whether it is true or false, and give a brief reason (or evidence) for your answer.

1. The order of a numerical method for solving an IVP for an ODE is the same as the order of its local truncation error.
2. The differential equation $y'(t) = f(t, y(t))$ is stable if $(\partial f/\partial z)(t, z)$ is nonpositive for all t and z.
3. Euler's method consists of forward- and backward Euler steps.
4. The global error of a numerical method for solving an IVP for an ODE is the sum of the local truncation errors.
5. When Euler's method is applied to the particular ODE $y'(t) = g(t)$ with initial condition $y(a) = 0$, and with $t_i = a + (i-1)h$, $i = 1{:}N$, and $h := (b-a)/(N-1)$, then y_N equals the result of the composite trapezoidal rule with stepsize h applied to $\int_a^b g(t)\,\mathrm{d}t$.