Divided difference table

x	$f[]$	$f[,]$	$f[,,]$	$f[,,,]$
$\mathbf{x}_0$	$f(x_0)$			
		$f[\mathbf{x}_0, x_1]$		
x_1	$f(x_1)$		$f[\mathbf{x}_0, x_1, \mathbf{x}_2]$	
		$f[x_1, \mathbf{x}_2]$		$f[x_0, x_1, x_2, x_3]$
$\mathbf{x}_2$	$f(x_2)$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f(x_3)$			

Note that the simple pattern

$$\frac{\texttt{below} - \texttt{above}}{x_{\texttt{below}} - x_{\texttt{above}}},$$

outlined for the particular case

$$\frac{f[x_1, \mathbf{x}_2] - f[\mathbf{x}_0, x_1]}{\mathbf{x}_2 - \mathbf{x}_0} = f[\mathbf{x}_0, x_1, \mathbf{x}_2],$$

is used to compute *all* the entries, with `below` and `above` the two left neighbors and, $x_{\texttt{below}}$ and $x_{\texttt{above}}$ the two *corresponding* x_j's that mark the basis of the triangle whose apex is the entry being computed.

What if $x_{\texttt{below}} = x_{\texttt{above}}$?? Then, assuming you have been wise enough to group all the x_i that equal $x_{\texttt{below}}$ together, the task now is to supply a divided difference $f[x_i, \ldots, x_j]$ with $x_i = x_{i+1} = \cdots = x_j$. Where to find it? Remember that, always, $f[x_i, \ldots, x_j] = f^{(j-i)}(\eta)/(j-i)!$ for some η in the smallest interval containing $x_i, \ldots, x_j$. So, if $x_i = \cdots = x_j$, then that smallest interval consists of just one point, hence $\eta = x_i$ in that case. So, in this case, we don't compute $f[x_i, \ldots, x_j]$ *from* the table, but (as in the case $i = j$), must supply it *to* the table from our knowledge of the function f or as data we want our interpolating polynomial to match.

Here is an illustration. Look at the divided difference table for the function $f(x) = x^2$, at the points (0,1,2,3), and also at the points (2,0,3,1):

x	$f[]$	$f[,]$	$f[,,]$	$f[,,,]$
0	0			
		1		
1	1		1	
		3		0
2	4		1	
		5		
3	9			

x	$f[]$	$f[,]$	$f[,,]$	$f[,,,]$
2	4			
		2		
0	0		1	
		3		0
3	9		1	
		4		
1	1			

We see that, indeed, $f[0, 1, 2, 3] = 0 = f[2, 0, 3, 1]$.

An example involving repeated points is worked out in the notes on cubic Hermite interpolation.