Divided difference table

$$\begin{array}{lllll}
x & f[\,] & f[,] & f[,,] & f[,,,] \\
\mathbf{x}_0 & f(x_0) & & & \\
 & & f[\mathbf{x}_0, x_1] & & \\
x_1 & f(x_1) & & f[\mathbf{x}_0, x_1, \mathbf{x}_2] & \\
 & & f[x_1, \mathbf{x}_2] & & f[x_0, x_1, x_2, x_3] \\
\mathbf{x}_2 & f(x_2) & & f[x_1, x_2, x_3] & \\
 & & f[x_2, x_3] & & \\
x_3 & f(x_3) & & &
\end{array}$$

Note that the simple pattern

$$\frac{\texttt{below} - \texttt{above}}{x_{\texttt{below}} - x_{\texttt{above}}},$$

outlined for the particular case

$$\frac{f[x_1, \mathbf{x}_2] - f[\mathbf{x}_0, x_1]}{\mathbf{x}_2 - \mathbf{x}_0} = f[\mathbf{x}_0, x_1, \mathbf{x}_2],$$

is used to compute *all* the entries, with `below` and `above` the two left neighbors and, $x_{\texttt{below}}$ and $x_{\texttt{above}}$ the two *corresponding* x_j's that mark the basis of the triangle whose apex is the entry being computed.

What if $x_{\texttt{below}} = x_{\texttt{above}}$?? Then, assuming you have been wise enough to group all the x_i that equal $x_{\texttt{below}}$ together, the task now is to supply a divided difference $f[x_i, \ldots, x_j]$ with $x_i = x_{i+1} = \cdots = x_j$. Where to find it? Remember that, always, $f[x_i, \ldots, x_j] = f^{(j-i)}(\eta)/(j-i)!$ for some η in the smallest interval containing $x_i, \ldots, x_j$. So, if $x_i = \cdots = x_j$, then that smallest interval consists of just one point, hence $\eta = x_i$ in that case. So, in this case, we don't compute $f[x_i, \ldots, x_j]$ *from* the table, but (as in the case $i = j$), must supply it *to* the table from our knowledge of the function f or as data we want our interpolating polynomial to match.

Here is an illustration. Look at the divided difference table for the function $f(x) = x^2$, at the points (0,1,2,3), and also at the points (2,0,3,1):

$$\begin{array}{lllll}
x & f[\,] & f[,] & f[,,] & f[,,,] \\
0 & 0 & & & \\
 & & 1 & & \\
1 & 1 & & 1 & \\
 & & 3 & & 0 \\
2 & 4 & & 1 & \\
 & & 5 & & \\
3 & 9 & & &
\end{array}
\qquad
\begin{array}{lllll}
x & f[\,] & f[,] & f[,,] & f[,,,] \\
2 & 4 & & & \\
 & & 2 & & \\
0 & 0 & & 1 & \\
 & & 3 & & 0 \\
3 & 9 & & 1 & \\
 & & 4 & & \\
1 & 1 & & &
\end{array}$$

We see that, indeed, $f[0, 1, 2, 3] = 0 = f[2, 0, 3, 1]$.

An example involving repeated points is worked out in the notes on cubic Hermite interpolation.

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