Answers to almost all problems in Chapter 1 (22sep02)

1.1 irrational numbers

1.2 purely imaginary numbers

1.3 (a) $\{-2,1\}$; (b) $\{1\}$; (c) the odd integers; (d) $-\mathbb{R}_+$.

1.4 4

1.5 (a) On a seating chart, showing the two-dimensional arrangement of seats and, on each seat, I'd write a student's name, if any.

(b) domain: the students in class; range: the occupied seats in class; target: all available seats in class.

1.6 (i) no; (ii) yes; (iii) no; (iv) no; (v) yes; (vi) no.

1.7 (a) $(0,0,0)$; (b) $(0,1,0)$; (c) $(0,0,1,0)$; (d) 0; (e) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; (f) same as (e).

1.8 a major effort

1.9 A

1.10 (a) $\mathbb{R}^{1\times 3}$; (b) $\mathbb{R}^{2\times 1}$; (c) $\mathbb{R}^{3\times 1}$; (d) $\{[2,3]\}$; (e) $\{3\}$; (f) $[-1\,..\,1]^{2\times 3}$; (g) $\{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\}$; (h) $\mathbb{R}^{1\times 3}$;

1.11 (a) $\max(\chi_R(t), \chi_S(t)) = 1$ iff $\chi_R(t) = 1$ or $\chi_S(t) = 1$ iff $t \in R$ or $t \in S$ iff $t \in R \cup S$.

(b) $\min(\chi_R(t), \chi_S(t)) = 1$ iff $\chi_R(t) = 1$ and $\chi_S(t) = 1$ iff $t \in R$ and $t \in S$ iff $t \in R \cap S$. Also, $\chi_R(t)\chi_S(t) = 1$ iff $\chi_R(t) = 1$ and $\chi_S(t) = 1$.

(c) $\chi_{T\backslash S} = 1 - \chi_S$, while $T\backslash S = T \cap (T\backslash S)$; now use (b).

(d) $R \subset S$ iff $t \in R \Longrightarrow t \in S$ iff $\chi_R(t) = 1 \Longrightarrow \chi_S(t) = 1$ iff $\chi_R \le \chi_S$.

1.12 $t \in f^{-1}(R \cup S)$ iff $f(t) \in R$ or $f(t) \in S$ iff $t \in f^{-1}R$ or $t \in f^{-1}S$ iff $t \in (f^{-1}R) \cup (f^{-1}S$. Etc.

1.13 If g were 1-1, then since, by definition of $\#Y$, for $m := \#Y$, there is $f : \underline{m} \to Y$ onto, (1.2)Lemma would imply that $n \le m$, a contradiction to $n > \#Y$.

1.14 If f were onto, then since, by definition of $\#Y$, for $n := \#Y$, there is $g : \underline{n} \to Y$ 1-1, (1.2)Lemma would imply that $n \le m$, a contradiction to $m < \#Y$.

1.15 (1.2)

1.16 By Problem 1.15, N is bounded since any 1-1 map into S gives rise to a 1-1 map into T; also N is not empty since it contains 0 (as the empty map into T surely is 1-1). Hence, there is a largest n such that some $g : \underline{n} \to S$ is 1-1. If $\operatorname{ran} g \ne S$, then, for some $s \in S$, the list $(g(1), \ldots, g(n), s)$ is still 1-1 and into S, contradicting the maximality of n. Hence g must be onto, and therefore $\#S = n$.

1.17 If $S = T$, then certainly $\#S = \#T$, and so $\#S \le \#T$.

If $S \subset T$, then by Problem 1.16, the finiteness of T implies the existence of $n \in \mathbb{N} \cup \{0\}$ and of some 1-1 map g on $\underline{n}$ onto S. Taking this as a 1-1 map into T, a previous homework permits us to extend this to a 1-1 map on some $\underline{m}$ onto T, with $\#S = n \le m = \#T$, and the added entries making up $T\backslash S$. In particular, $n = m$ if and only if $T\backslash S = \{\}$, i.e., $S = T$.

1.18 all yours

1.19 (a) yes, but neither 1-1 nor onto; (b) no; (c) yes, and onto but not 1-1; (d) yes, and 1-1 but not onto.

1.20 (a) no; (b) yes, and 1-1 but not onto; (c) no; (d) no.

For R^{-1} to be the graph of a map, R itself must be onto (if it is a map). Also, if R is 1-1 and onto, then so is R^{-1}.

1.21 (a) $fg : \underline{3} \to \underline{3}$ with list $(3,2,3)$; $gf : \underline{2} \to \underline{2}$ with list $(1,2)$, i.e., the identity.

(b) $\operatorname{ran}(fg) = \{2,3\}$ and, for $j = 2,3$, we have, e.g., from (a), that $(fg)(j) = j$.

1.22 (a) list has 3 entries, hence dom is $\underline{3} =$ tar, hence map is identity, trivially invertible. (b) Now, tar $= \underline{4} \ne \underline{3} =$ dom; map is 1-1 but not onto; list for a left inverse is $(1,2,3,1)$. (c) dom $= \underline{3} \ne \underline{2} =$ tar, map is not 1-1, but onto; list for a right inverse is $(1,2)$. (d) note that $dh : \mathbb{R} \to \mathbb{R} : y \mapsto d(h(y)) = d(y/2, 0) = 2*(y/2) - 3*0 = y$ is the identity, hence d is onto, with h as right inverse. But d is not 1-1 since it maps all vectors $(3*y, -2*y) : y \in \mathbb{R})$ to 0. (e) Notice that f rotates the plane 90 degrees counterclockwise, hence invertible, with the inverse the rotation 90 degrees clockwise, i.e., $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto (x_2, -x_1)$. Indeed, $f(x_2, -x_1) = (-(-x_1), x_2) = x$ and $f^{-1}(-x_2, x_1) = (x_1, -(-x_2) = x$. (f) g is translation by the vector $(2, -3)$, hence translation by $-(2, -3)$ is its inverse. (g) From (d), we know that d is a left inverse to h; in particular, h is 1-1; it is obviously not onto, since everything in its range has 0 second component.

1.23 Already verified in (d) above that $dh =$ id. $hd : \mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto h(2x_1 - 3x_2) = (xbutnot - (3/2)x_2, 0)$ maps $\mathbb{R}^2$ to the first axis in the plane, but leaves the axis itself fixed.

1.24 g,h have same domain and target. Also, for any $s \in S$, $f(g(s)) = (fg)(s) = (fh)(s) = f(h(s))$, while f is 1-1, therefore also $g(s) = h(s)$.

1.25 f, g have the same domain and target. Also, for any $t \in T$, since h is onto, there is $s \in S$ with $t = h(s)$, hence $f(t) = fh(s) = gh(s) = g(t)$.

1.26 (a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 1 \\ 3 & 4 & 1 & 2 & 2 \\ 4 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 2 & 3 & 4 & 1 & 1 \end{bmatrix}$, so $f^5 = f^1$, i.e., $d = 4$ and $f^d = (1, 2, 3, 4, 4)$.

(b) (A) $d = 5$, $f^d =$ id. (B) $d = 6$, $f^d =$ id. (C) $d = 1$, $f^d =$ id. (D) $d = 3$, $f^d = (1, 2, 1, 1, 5)$. (E) $d = 2$, $f^d = (5, 2, 5, 2, 5)$. (F) $d = 1$, $f^d = (2, 5, 2, 2, 5)$.

(c) $f^d =$ id in case f is invertible.

1.27 [c, b] = sort(a); if any(c =(1:length(a))), fprintf('The input doesn''t describe an invertible map'), b = []; end

1.28 Since we know maps that have a right or a left inverse without being invertible, we know that part of this problem is wrong, i.e., that g can be invertible without the f_i being invertible.

On the other hand, if all the f_i are invertible, then one verifies directly that $f_n^{-1} \cdots f_1^{-1}(f_1 \cdots f_n) = \text{id} = (f_1 \cdots f_n)f_n^{-1} \cdots f_1^{-1} = \text{id}$.

1.29 Not necessarily; e.g., if $\#S = 1$, then there is exactly one $g : T \to S$, and it is the unique left inverse for every $f : S \to T$, yet none is invertible unless $\#T = 1$. (However, the converse does hold if $\#S > 1$.)

1.30 $gf = \text{id}_S$ by assumption, hence g is onto. If now also g is 1-1, then it is invertible, and then f, being a right inverse for g, must be its inverse and, in particular, invertible, hence g, being a left inverse for it, must be its unique left inverse.

If g fails to be 1-1, then, since the identity $gf = \text{id}_S$ forces it to be 1-1 on $\operatorname{ran} f$, there must be some $s \in S$ which, in addition to $f(s)$, has some $t \in T \backslash \operatorname{ran} f$ as a pre-image under g. Since $\#S > 1$, we can obtain from g a different left inverse, g_1, by having it coincide with g off t, and setting $g_1(t)$ to some value other than s.

1.31 Since $fg = \text{id}_T$ by assumption, g is 1-1. Hence if it is also onto, then g is invertible, and then f, being a left inverse for it, must be its inverse, hence itself invertible and therefore has exactly one right inverse, necessarily its inverse, g.

Assume that g is not onto, hence there is $s \in S \backslash g(T)$. Since $fg = \text{id}_T$, there is $s_1 \in g(T)$ with $f(s_1) = f(s)$. But now, constructing g_1 to agree with g off $f(s) = f(s_1)$, but taking the value $s_1 \neq s$ at $f(s) = f(s_1)$ makes also $g_1 \neq g$ a right inverse for f.

1.32 Yes. If g is a right inverse for f, then $fg = \text{id}_T$, hence f is onto. If now f is not invertible, then it must fail to be 1-1, i.e., $f(s_1) = f(s_2)$ for some $s_1 \neq s_2$, hence, with $t_j \in T$ such that $s_j = g(t_j)$, $j = 1{:}2$, the equation $fg = \text{id}_T$ is unchanged if we change g to map t_j to s_{3-j}, hence the resulting different map is also a right inverse for f.

1.33 (i) fg onto $\Longrightarrow$ f onto (for any g into X). Conversely, since our g is onto, having f onto $\Longrightarrow$ fg

onto (as the composition of onto maps).

fg 1-1 $\Longrightarrow f = (fg)g^{-1}$ is 1-1, while f 1-1 implies that fg is 1-1 (both times as the composition of 1-1 maps). Note that we have only used that g is invertible; the finiteness of X played no role so far.

(ii) Directly, i.e., without use of (i), if $f : X \to Y$ with $n := \#X = \#Y < \infty$, then there exist invertible $g : \underline{n} \to X$ and $h : \underline{n} \to Y$. If now f is 1-1(onto), then $h^{-1}fg : \underline{n} \to \underline{n}$ is 1-1(onto), hence, by (1.3), invertible, and that makes $f = h(h^{-1}fg)g^{-1}$ invertible, too.

1.34 (a)F; (b)T; (c)T; (d)T; (e)T; (f)F; (g)F; (h)T; (i)T; (j)T; (k)F; (l)T; (m)F.