Answers to all problems in chapter 10

10.1 (i) Many ways to handle this. E.g., notice that the first column of A is bound and the second is free. Hence $\operatorname{ran} A = \operatorname{ran} A(:,1)$, therefore, in particular, (1) A must map $A_{:1} = (1,2)$ to a multiple of itself and, indeed, $A_{1,2} = (5,10) = 5(1,2)$; and (2) $(2,-1) \in \operatorname{null} A$, hence an eigenvector for the eigenvalue 0.

So, $A = V\mathrm{M}V^{-1}$ with $V = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $\mathrm{M} = \operatorname{diag}(5,0)$. Then $V^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}/5$.

(ii) $\exp(A) = V \exp(\operatorname{diag}(5,0)) V^{-1} = (\begin{bmatrix} 1 \\ 2 \end{bmatrix} \exp(5) \, [1 \quad 2] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \exp(0) \, [2 \quad -1])/5 = \begin{bmatrix} E+4 & 2E-2 \\ 2E-2 & 4E+1 \end{bmatrix}/5$ with $E := \exp(5) = 148.4132$, or $\begin{bmatrix} 30.4826 & 58.9653 \\ 58.9653 & 118.9305 \end{bmatrix}$.

10.2 $\mu = 3$: $A - 3\,\mathrm{id} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ and, by inspection, on first column is bound, hence $\operatorname{rrref}(A - 3\,\mathrm{id}) = [1 \quad 1 \quad -1]$, therefore, by recipe, a basis for $\operatorname{null}(A - 3\,\mathrm{id})$ is $\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\mu = 5$: $A - 5\,\mathrm{id} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$, leading to $\operatorname{rrref}(A - 5\,\mathrm{id}) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$, and the recipe for a basis for $\operatorname{null}(A - 5\,\mathrm{id})$ gives $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

10.3 (a) $[V, AV] = \begin{bmatrix} 0 & 2 & 2 & 8 \\ 3 & 1 & 13 & 7 \\ 1 & 1 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 2 & 8 \\ 0 & -2 & -2 & -8 \\ 1 & 1 & 5 & 5 \end{bmatrix}$ from which we see that columns 3 and 4 are free, hence $\operatorname{ran} AV \subset \operatorname{ran} V$.

(b) Continuing one step further, we get the equivalent matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 4 & 1 \end{bmatrix} =: M[V, AV]$ for some invertible matrix M. This shows that $M([3,2],:)V = \mathrm{id}_2$, hence $M([3,2],:)$ is a left inverse for V and so, in particular $C := \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = M([3,2],:)AV$ is the sought-for matrix representation of $B = A|_Y$ wrto the basis V of Y.

Therefore, $\operatorname{spec}(B) = \operatorname{spec}(C)$. Notice that C maps $(1,1)$ to $(5,5)$, and maps $(1,-1)$ to $(3,-3)$, hence $\operatorname{spec}(B) = \{5, -3\}$.

10.4 Since A is upper triangular, yet has only 0 on its diagonal, we know from (3.19)Proposition that $\operatorname{spec}(A) = \{0\}$. Also, since the first column is the only free one, $\operatorname{null}(A - 0\,\mathrm{id})$ is 1-dimensional, and e_1 is obviously in it, hence $[e_1]$ is a basis for it, i.e., up to scalar multiples, it is the only eigenvector for A.

10.5 (i) $Ax = \mu x$ implies $(\alpha A)x = \alpha(Ax) = \alpha\mu x$. (ii)$(A + \alpha\,\mathrm{id})x = Ax + \alpha x = (\mu + \alpha)x$. (iii) By induction on k since $A^0 x = x = \mu^0 x$, if we already know for some k that $A^k x = \mu^k x$, then also $A^{k+1}x = A(A^k x) = A(\mu^k x) = \mu^{k+1}x$. (iv) A invertible implies that $\operatorname{null}(A) = \{0\}$, hence $\mu \neq 0$, while $x = A^{-1}Ax = A^{-1}(\mu x)$, hence $A^{-1}x = (1/\mu)x$. (v) With B' either B^{t} or B^{c}, we know that $\operatorname{rank} B' = \operatorname{rank} B$, hence if B is also square, then also $\dim \operatorname{null} B' = \dim \operatorname{null} B$. In particular, since we know that $\operatorname{null}(A - \mu\,\mathrm{id})$ is not trivial, we also know that $\operatorname{null}(A^{\mathrm{t}} - \mu\,\mathrm{id})$ and $\operatorname{null}(A^{\mathrm{c}} - \overline{\mu}\,\mathrm{id})$ are not trivial, hence, μ is an eigenvalue for A^{t} and $\overline{\mu}$ is an eigenvalue for A^{c}.

10.6 By diagonalizability, there is basis V for X consisting of eigenvectors for A, i.e., $Av_j = \mu_j v_j$, all j. $\#\operatorname{spec}(A) = 1$ says that all the μ_j are the same scalar, μ say. Hence $AV = \mu V = (\mu\,\mathrm{id}_X)V$. Since V is a basis, this shows that $A = \mu\,\mathrm{id}_X$.

10.7 $A = \operatorname{diag}(1,2,3)$.

10.8 (a) No ($\operatorname{ran} A \cap \operatorname{null} A = \{0\} \cap \operatorname{null} A = \{0\}$); (b) No ($\operatorname{ran} A = \operatorname{ran} A(:,1)$ while $\operatorname{null} A = \operatorname{ran} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, hence $\operatorname{ran} A \perp \operatorname{null} A$): (c) Yes ($(-1,2)$ is in both $\operatorname{ran} A$ and $\operatorname{null} A$); (d) No (since 0 is not even an eigenvalue, let alone a defective one).

10.9 Since $X = \operatorname{ran} P \oplus \operatorname{null} p$, (4.26) implies that $[U, W]$ is a basis for X. Moreover, since P is a linear projector, $\operatorname{ran} P = \{x \in X : Px = x\}$, hence $PU = U$, while certainly $PW = 0$. Hence, $[U, W]$ is an eigenbasis for P.

10.10 (i)using $x = e_1$, we get $[x, Ax, A^2x] = \begin{bmatrix} 1 & 7 & 29 \\ 0 & 5 & 25 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \end{bmatrix}$, so $p(t) = t^2 - 5t + 6 = (t-3)(t-2)$, giving $\mu = 3$, with corresponding eigenvector $q(A)x$ with $q(t) = p(t)/(t-3) = t-2$, i.e., $A_{:1} - 2e_1 = (5,5)$. The other eigenvalue is $\mu = 2$, with corresponding eigenvector $A_{:1} - 3e_1 = (4,5)$.

(ii) With $x = e_3$, get $[x = e_3, Ax = e_2, A^2x = e_1, A^3x = 0]$, hence the fourth column is the first free one, with $(0,0,0,1)$ the corresponding element in the nullspace. Hence, the minimal polynomial for A at e_3 is $p(t) = t^3$, and it has only one zero, namely $\mu = 0$, and $q(t) := p(t)/(t-0) = t^2$, hence the corresponding eigenvector is $A^2e_3 = unitv1$.

(iii) With $x = unitv1$, we get $B := [x, Ax, A^2x, A^3x] = \begin{bmatrix} 1 & -1 & 15 & 175 \\ 0 & 20 & 100 & 500 \\ 0 & 2 & -30 & -350 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 20 & 200 \\ 0 & 1 & 5 & 25 \\ 0 & 0 & -40 & -400 \end{bmatrix} \to$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -25 \\ 0 & 0 & 1 & 10 \end{bmatrix}$, hence an element in the nullspace is $(0, 25, -10, 1)$, giving the minimal polynomial at $x = e_1$ of $p(t) = t^3 - 10t^2 + 25t = (t-5)^2 t$. For $\mu = 5$, we get as eigenvector $(A-5)Ae_1 = B(:,3) - 5B(:,2) = (20, 0, -40)$, or, simpler, $(1, 0, -2)$. For $\mu = 0$, get the eigenvector $(A-5)^2 e_1 = B_{:3} - 10B_{:2} + 25B_{:1} = (50, -100, -50)$ or, simpler, $(1, -2, -1)$.

10.11 (a) If $Ax = \mu x$, then we verified earlier that $A^k x = \mu^k x$ for all $k = 0, 1, 2, \ldots$, hence, with $p =: a_0 + a_1()^1 + \cdots + a_k()^k$, we have $p(A)x = (a_0 + a_1\mu + \cdots + a_k\mu^k)x = p(\mu)x$. Hence, if $p(A) = 0$, then, for any $\mu \in \operatorname{spec}(A)$ with corresponding eigenvector x, we have $0 = p(A)x = p(\mu)x$ which implies $p(\mu) = 0$ since $x \neq 0$. (b) Then $p(A) = 0$ with $p = ()^2 - ()^1 = ()^1(()^1 - 1)$, therefore $\operatorname{spec}(A) \subset \{0, 1\}$. (c) Then $p(A) = 0$ for $p = ()^q$, therefore $\operatorname{spec}(A) \subset \{0\}$. (d) Since D^{k+1} maps Π_k to $\{0\}$, (c) implies that $\operatorname{spec}(A) \subset \{h0\}$, while the fact that $D()^0 = 0$ implies that 0 is an eigenvalue for D. Hence $\operatorname{spec}(D : \Pi_k \to \Pi_k) = \{0\}$.

10.12 (i) $[e_1, Ae_1, A^2e_1, \ldots] = [e_1, e_2, 0, \ldots]$ gives $()^2$ as the minimal polynomial. Since it is of degree 2 = order of A, its zero set is the spectrum; hence $\operatorname{spec}(A) = \{0\}$.

(ii) $[e_1, Ae_1, A^2e_1, \ldots] = [e_1, e_2, e_3, e_1, \ldots]$, so the minimal polynomial at e_1 is $()^3 - 1$. Since its degree equals the order of A, its zeroset is the spectrum, hence $\operatorname{spec}(A) = \{\exp(j2\pi \mathrm{i}) : j = 1{:}3\}$.

(iii) $[e_1, Ae_1, \ldots] = [e_1, e_2, e_2, \ldots]$, so the min.annil.pol. at e_1 is $()^2 - ()$, of degree equal to the order of A, hence $\operatorname{spec}(()A) = \{0, 1\}$.

(iv) $[e_1, Ae_1, \ldots] = [e_1, e_2, e_1, \ldots]$, so the min.annil.pol. at e_1 is $()^2 - 1$, showing that $\{1, -1\} \subset \operatorname{spec}(A)$. But, by inspection, $Ae_3 = 2e_3$, hence also $3 \in \operatorname{spec}(A)$. Since A is of order 3, must have $\#\operatorname{spec}(A) \leq 3$, hence conclude that $\operatorname{spec}(A) = \{-1, 1, 2\}$.

10.13 (i) If $x \in \operatorname{null} A \cap \operatorname{null} B$, then $Ax = 0 = Bx$, hence also $(A+B)x = 0$.

(ii) If $x \in \operatorname{null} A + \operatorname{null} B$, then $x = y + z$ with $y \operatorname{null} A$ and $z \in \operatorname{null} B$. Hence $ABx = AB(y+z) = ABy + ABz = BAy + 0 = 0 + 0 = 0$.

(iii) Let $q_j := p_j/d$, all j. By assumption, the q_j have no common divisor, hence there exist h_j so that $\sum(h_j q_j) = 1$???

10.14 Since Y is D-invariant, the restriction $D|_Y$ of D to Y is a linear map on Y. Since Y is finite-dimensional, $D|_Y$ has a minimal annihilating polynomial, p say, and $\deg p \leq \dim Y$. This implies that $Y \subset \operatorname{null} p(D)$, therefore, since $\dim \operatorname{null} p(D) = \deg p$, $Y = \operatorname{null} p(D)$.

It follows that Y is spanned by certain **exponential polynomial**s, i.e., functions of the form $t \mapsto q(t)\exp(\xi t)$ for certain polynomials q and scalars ξ, the latter being the roots of p.

10.15 (a)T; (b)T; (c)F; (d)F; (e)T; (f)F; (g)T; (h)T; (i)T; (j)T; (k)F.