Answers to some problems in chapter 11

11.1 (i) $A^k = \alpha^k \,\mathrm{id}_X$; this converges iff (α^k) converges; by Backgrounder, iff $|\alpha| \le 1$, with equality iff $\alpha = 1$.

(ii) $A^k = \begin{bmatrix} 2^{-k} & 2^{10}2^{-k+1}k \\ 0 & 2^{-k} \end{bmatrix} \to 0$.

(iii) $A^{2k} = \mathrm{id}_2$, $A^{2k+1} = A \ne \mathrm{id}_2$, hence, no convergence.

(iv) $A^k = \begin{bmatrix} a^k & b\sum_{i+j=k-1} a^i c^j \\ 0 & c^k \end{bmatrix}$; this converges iff (both (a^k) and (c^k) converge, i.e., $|a| \le 1$ with equality only if $a = 1$, and $|c| \le 1$ with equality only if $c = 1$), AND, if $a = 1 = c$, then $b = 0$; limit is 0 if $|a|, |c| < 1$; $\begin{bmatrix} 1 & b/(1-c) \\ 0 & 0 \end{bmatrix}$ if $a = 1 > |c|$; $\begin{bmatrix} 0 & b/(1-a) \\ 0 & 1 \end{bmatrix}$ if $|a| < 1 = c$; id_2 if $a = 1 = c$ and $b = 0$.

11.2 Since (c) implies (b) implies (a), I'll only give the extreme property. (i) (b) $(((\mathrm{id}_n)^k : k = 1, 2, \ldots)$ is a constant sequence); (ii) not even (a) (since $[1, 1; 0, 1]^k = [1, k; 0, 1]$); (iii) (c) (since $\mathrm{spec}() = \{8/9\}$, hence spectral radius is < 1); (iv) (a).

11.3 (i) Assume that $Aa = 0$. Then, writing $a = (b, c)$ appropriately, we have $0 = Aa = (Bb, Cb + Dc)$. Hence, if B is invertible, then b must be zero, therefore $0 = Cb + Dc = Dc$ and now, if also D is invertible, then also $c = 0$, hence altogether $a = 0$, showing the square matrix A to be invertible. Conversely, if B is not invertible, then there is some nonzero b with $Bb = 0$, then $a = (b, 0)$ is a nonzero vector with $Aa = 0$. Again, if D is not invertible, then there is some nonzero c with $Dc = 0$ and then $a = (0, c)$ is a nonzero vector with $Aa = 0$.

Since, for any $\mu \in \mathbb{F}$, $(A - \mu\,\mathrm{id}) = \begin{bmatrix} B - \mu\,\mathrm{id} & C \\ 0 & D - \mu\,\mathrm{id} \end{bmatrix}$, it follows that $\mu \in \mathrm{spec}(A)$ if and only if $\mu \in \mathrm{spec}(B)$ or $\mu \in \mathrm{spec}(C)$.

11.4 By previous homework, $\mathrm{spec}(A) = \mathrm{spec}(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}) \cup \mathrm{spec}(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}) = \{0, 5\} \cup \{-1, 3\}$, the last equality by earlier homework and (10.21)Example.

11.5 By previous homework, $\mathrm{spec}(A) = \{-1, 3\} \cup \{3\} = \{-1, 3\}$.

For $\mu = -1$, $A - \mu\,\mathrm{id} = \begin{bmatrix} 2 & 2 & a \\ 2 & 2 & b \\ 0 & 0 & 4 \end{bmatrix}$, hence the only free column is the second one, hence regardless of our choice of a and b, $\mathrm{null}(A - \mu\,\mathrm{id})$ has $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a basis, but, in $\begin{bmatrix} 2 & 2 & a & 1 \\ 2 & 2 & b & -1 \\ 0 & 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & a & 1 \\ 0 & 0 & b-a & -2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$, the last column is bound, hence not in $\mathrm{ran}(A - \mu\,\mathrm{id})$. So, -1 is not defective.

For $\mu = 3$, $A - \mu\,\mathrm{id} = \begin{bmatrix} -2 & 2 & a \\ 2 & -2 & b \\ 0 & 0 & 0 \end{bmatrix}$, hence the second column is free, while the third column is free if and only if $a + b = 0$. So, if $a + b \ne 0$, then $\mathrm{ran}(A - \mu\,\mathrm{id}) = \mathrm{ran}[e_1, e_2]$, hence contains the eigenvector $(1, 1, 0)$ belonging to $\mu = 3$, i.e., 3 is defective. If $a + b = 0$, then $\mathrm{ran}(A - \mu\,\mathrm{id}) = \mathrm{ran}[v]$ with $v := (1, -1, 0)$ evidently not in $\mathrm{null}(A - \mu\,\mathrm{id})$, hence 3 is not defective in this case.

11.6 (a) Calculations like the following:

```
n = 10; ax = repmat(x,1,n);
for j=2:n, ax(:,j)=A*ax(:,j-1); end
ratios = ax(:,2:n)./ax(:,1:n-1)
```

show convergence to what `eig(A)` says is the unique absolutely largest of four eigenvalues, hence a nondefective eval, namely 1. In the later steps, error is at each step cut in half, confirmed by the absolutely largest eigenvalue being around .41 in absolute value.

(b) By inspection, the iteration cycles through the vectors (1,-1), -(1,1), (-1,1), (1,1), (1,-1), ..., hence no convergence; the reason: there are two eigenvalues of maximum absolute value.

(c) By inspection, the iterates are (1,k), k=1,2, ..., hence the two ratios are 1 and (k+1)/k, the second also converging to 1, but the error reduction is slower and slower; the reason: the only eigenvalue is defective.

(d) $Ax = 3x$, i.e., x is an eigenvector, hence instant convergence. But, according to `eig(A)`, 3 is not the absolutely largest eigenvalue; the reason: the x proposed happens to have no component in the direction of the eigenvector belonging to the absolutely largest eigenvalue.

11.7 (a) T; (b) T; (c) F; (d) F; (e) F.