Answers to (almost?) all problems in chapter 2 (30sep02)

2.1 (a) no: not closed under addition; (b) yes: nonempty and, since vector ops are pointwise, the condition $x_1 = x_2$ is preserved under them, closed under addition and scalar mult.; (c) no: not closed under mult. by *negative* scalar; (d) yes: the trivial subspace of $\mathbb{R}^3$; (e) no: it's empty; (f) yes: nonempty (e.g., contains $0 : [a \,..\, b] \to \mathbb{R} : t \mapsto 0$), and, since vector ops are pointwise, and $\lim_{s \to t}(\alpha f(s) + g(s))$ exists and equals $\alpha \lim_{s \to t} f(s) + \lim_{s \to t} g(s)$ when the latter two limits exist, $C[a \,..\, 2]$ is closed under vector ops. (g) yes: nonempty (e.g., contains $0 \in \mathbb{R}^{3 \times 3}$) and the vector ops are pointwise, hence preserve the condition that certain entries are zero.

2.2 $0x + 0x = (0 + 0)x = 0x$ by (s.2), hence $0x = 0$ by (a.1) and (a.4). With that, $x + (-1)x = 1x + (-1)x = (1 - 1)x = 0x = 0$, by (s.4), (s.2), hence $(-1)x$ must be $-x$ by the uniqueness of the neutral element (claimed but not proved after the Definition; for completeness: if also $x + z = 0$, then, by (a.1), (a.2), (a.3), $-x = -x + 0 = -x + (x + z) = (-x + x) + z = 0 + z = z$).

2.3 Let $(Y_i : i \in I)$ be a collection of linear subspaces of the vector space X, and let $Z := \cap_{i \in I} Y_i$ be their intersection.

Since any subspace contains 0, so does Z, hence $Z \neq \{\}$.

If $x, y \in Z$, then, for every for every every $i \in I$, $x, y \in Y_i$, hence, for every for every $\alpha, \beta \in \mathbb{F}$, $\alpha x + \beta y \in Y_i$, therefore also in Z.

2.4 If one of them contains the other, then the union equals that one, hence is a linear subspace.

Conversely, if neither contains the other, then there is $y \in Y \backslash Z$ and $z \in Z \backslash Y$. Now having $y + z \in Y \cup Z$ would imply, wlog, that $y + z \in Y$, hence also $z = (y + z) - y \in Y$, a contradiction. Hence, $y + z$ fails to be in $Y \cup Z$, i.e., $Y \cup Z$ fails to be closed under addition.

2.5 Let Z be the linear subspace. Since a linear subspace is closed under the vector operations, they are, in particular defined when restricted to that subspace, and all the conditions that require equality between two expressions (i.e., (a.1-2) and (s.1-4)) continue to hold. This leaves

(a.3): Since Z is non-empty (by definition of linear subspace), there is $z \in Z$, hence also $0 = 0z$ must be in Z (since Z is closed under multiplication by any scalar), and 0 continues to function as the neutral element.

(a.4): If $z \in Z$, then also $-z = (-1)z \in Z$.

2.6 (a) not defined since the trivial polynomial has infinitely many zeros; in any case, $\mathbb{N}$ is not a linear space, nor is the map homogeneous nor is it additive.

(b) positive homogeneous, but not homogeneous nor additive.

(c) yes (special case of evaluation functional).

(d) yes (since vector operations on $L(X, Y)$ are defined pointwise).

(e)

(f) no (e.g., $\sin(\alpha x) \neq \alpha \sin(x)$).

2.7 Since each element of $\operatorname{ran} f$ is of the form $f(x)$ for some $x \in X$, (2.5) shows that we can sum arbitrary elements of $\operatorname{ran} f$ and multiply them by arbitrary scalars. With that, we can verify the eight conditions in (2.1) by knowing them to hold in X and then using (2.5) to transfer them to $\operatorname{ran} f$.

E.g., $(\alpha + \beta) f(x) = f((\alpha + \beta)x) = f(\alpha x + \beta x) = \alpha f(x) + \beta f(x)$, with the first and last equality by (2.5), and the second equality by (s.2) of (2.1), thus verifying (s.2) for $\operatorname{ran} f$ as well.

2.8 (i) Such rotation carries e_1 to $-e_2$ and e_2 to e_1; hence $A = [-e_2, e_1]$.

(ii) The hyperplane contains e_j for $j < n$, and these are all kept fixed, while reflecting e_n across this hyperplane carries it to its negative; so, $B = [e_1, \ldots, e_{n-1}, -e_n]$.

(iii) Same as (ii).

(iv) Keeping the y-axis fixed means, in particular, keeping e_2 fixed, hence $D = [?, e_2]$. The condition $(2, 1) = [?, e_2](1, 1) = ? + e_2$ implies that $? = 2e_1$, hence, $D = [2e_1, e_2]$.

2.9 (i) $A^2 = [e_1, 0][e_1, 0] = [e_1, 0] = A$; $AB = [e_1, 0][0, e_1] = [0, e_1] = B$; $BA = [0, e_1][e_1, 0] = [0, 0] = 0$; $B^2 = 0$.

(ii) $A^2 = \mathrm{id}_2$, $AB = [e_1, -e_2] = -BA$, $B^2 = -\mathrm{id}_2$.

(iii) $B = A^2 = [e_2, e_3, e_1][e_2, e_3, e_1] = [e_3, e_1, e_2]$; $BA = A^3 = AB = [e_2, e_3, e_1][e_3, e_1, e_2] = [e_1, e_2, e_3] = \mathrm{id}_3$; $B^2 = A^4 = A^3A = A$.

2.10 (a)$BA = A$ but AB makes no sense since $\operatorname{tar} B = \mathbb{R}^2 \neq \mathbb{R}^3 = \operatorname{dom} A$; (b)$AA^{\mathrm{t}} = \begin{bmatrix} 21 & 9 \\ 9 & 5 \end{bmatrix}$, $A^{\mathrm{t}}A = \begin{bmatrix} 4 & 2 & 8 \\ 2 & 2 & 6 \\ 8 & 6 & 20 \end{bmatrix}$; (c)$AB = \begin{bmatrix} -2 & 0 & 15 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{bmatrix}$, $BA = \begin{bmatrix} -2 & -2 & -8 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ (d)$AB = \begin{bmatrix} 16-7\mathrm{i} & 22+5\mathrm{i} & 5+4\mathrm{i} \\ 15-5\mathrm{i} & 21+7\mathrm{i} & 9+11\mathrm{i} \end{bmatrix}$, but BA makes no sense since $\operatorname{dom} B = \mathbb{C}^2 \neq \mathbb{C}^3 = \operatorname{tar} A$.

2.11 We are looking for matrices A and B of order 2 that don't commute. E.g., $A = [0e_1]$ and $B = A^{\mathrm{t}} = [e_2, 0]$ give $AB = [e_1, 0]$ and $BA = [0, e_2]$.

2.12 Each should have a nontrivial range, and that range should like in the nullspace of the other. Simplest: $A = B = [0, e_1]$.

2.13 This problem is messed up! Need also to assume that $\#C = \#D$, and only then can we form $[C; D]$ and sum AC and BD, and then get $(AC + BD)_{ij} = (AC)_{ij} + (BD)_{ij} = \sum_{k=1}^{\#A} A_{ik}C_{kj} + \sum_{\ell=1}^{\#B} B_{i\ell}D_{\ell j} = \sum_{k=1}^{\#A} [A, B]_{ik}[C; D]_{kj} + \sum_{\ell=1}^{\#B} [A, B]_{i,\#A+\ell}[C; D]_{\#A+\ell,j} = \sum_{k=1}^{\#[A,B]} [A, B]_{ik}[C; D]_{kj} = ([A, B][C; D])_{ij}$.

2.14 The linear map $\mathbb{R}^2 \to \mathbb{C} : a \mapsto a_1 + \mathrm{i}a_2$ is 1-1 and onto, and $\mathbb{C} \to \mathbb{R}^2 : z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$ is its inverse, hence linear; therefore, in particular, both Re and Im are linear.

2.15 (a) $A = \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}$, $y = (4, -6)$; (b) same as (a); (c) $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 3 & 0 \end{bmatrix}$, $y = (16, 9)$.

2.16 (a) $\begin{bmatrix} 2x_1 & - & 3x_2 & = & 9 \\ 6x_1 & - & 4x_2 & = & -\sqrt{3} \\ \mathrm{e}x_1 & & -2x_2 & = & 1 \end{bmatrix}$; (b) $\begin{bmatrix} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 10 \\ 4x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & 10 \end{bmatrix}$; (c)

2.17 Given that $\operatorname{dom}(AB) = \operatorname{dom}(B)$ and $\operatorname{tar}(AB) = \operatorname{tar}(B)$, the equality says that A and B have the same domain and target. Since they are matrices, this makes them square.

2.18 `ones(2)`

2.19 $A^{-1} = A/9$

2.20 Let $B := \begin{bmatrix} d & -b \\ -c & acr \end{bmatrix}$. Then $AB = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = BA$. If $ad = bc$, then $AB = 0$, hence either $B = 0$, but then also $A = 0$, or else $\operatorname{null} A \supset \operatorname{ran} B \neq \{0\}$, hence A is not 1-1. Either way, A is not invertible. If $ad \neq bc$, then $A(B/(ad - bc)) = \mathrm{id}_2 = (B/(ad - bc))A$, hence $A^{-1} = B/(ad - bc)$.

Since A is a square matrix, it invertibility already follows from having it 1-1 or onto, e.g., knowing that $AB = (ad - bc)\,\mathrm{id}_2$ with $ad \neq bc$ is already sufficient.

2.21 The target of this map is $\mathbb{R}^{2\times 2}$, not $\mathbb{R}^2$. With that correction, $f(\alpha(a + \mathrm{i}b) + (c + \mathrm{i}d)) = \begin{bmatrix} (\alpha a + c) & -(\alpha b + d) \\ \alpha b + d & \alpha a + c \end{bmatrix} = \alpha \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \alpha f(a + \mathrm{i}b) + f(c + \mathrm{i}d)$, proving linearity. With that, the fact that $f(z) = 0$ implies that $z = 0$ shows that f is 1-1.

2.22 see, e.g., Problem 2.9(i).

2.23 Since the invertibility of AB implies that of both A and B in case they are square matrices, the example must have nonsquare A, B, since such matrices are automatically not invertible. So, neither $A = [1, 0]$ nor $B = A^{\mathrm{t}} = [1; 0]$ is invertible, but $AB = \mathrm{id}_1$ is.

2.24 $[A, B; 0, C][A^{-1}, -A^{-1}BC^{-1}; 0, C^{-1}] = [AA^{-1}, A(-A^{-1}BC^{-1}) + BC^{-1}; 0, CC^{-1}] = [\mathrm{id}, 0; 0, \mathrm{id}] = \mathrm{id}$ verifies that the second factor is a right inverse, hence the inverse since $[A, B; 0, C]$ is square.

2.25 Since A is invertible, $A + yz^{\mathrm{t}} = A(\mathrm{id} + (A^{-1}y)z^{\mathrm{t}})$ is invertible iff its right factor is and, by (2.19), that factor is invertible iff $\alpha \neq 0$, in which case

$$(\mathrm{id} + A^{-1}yz^{\mathrm{t}})^{-1} = \mathrm{id} - \alpha^{-1}A^{-1}yz^{\mathrm{t}},$$

and the formula follows by multiplying this from the right by A^{-1}.

2.26 generalize (2.19): $(\mathrm{id} + CD^{\mathrm{t}})(\mathrm{id} + CED^{\mathrm{t}}) = \mathrm{id} + CED^{\mathrm{t}} + CD^{\mathrm{t}} + CD^{\mathrm{t}}CED^{\mathrm{t}} = \mathrm{id} + C(E + \mathrm{id} + D^{\mathrm{t}}CE)$, hence if $E = -(\mathrm{id} + D^{\mathrm{t}}C)^{-1}$, then $\mathrm{id} + CED^{\mathrm{t}}$ is a right inverse for $\mathrm{id} + CD^{\mathrm{t}}$, hence an inverse since $\mathrm{id} + CD^{\mathrm{t}}$ is a square matrix. Now apply this to the second factor in the factorization $(A + CD^{\mathrm{t}}) = A(\mathrm{id} + A^{-1}CD^{\mathrm{t}})$.

2.27 (a)F (e.g., take $B = -A$); (b)F (e.g., $A = B = [0, e_1]$); (c)T; (d)T (by induction on n); (e)T (then $0 = (A^{\mathrm{t}}A)_{ii} = \sum_i (A_{ij})^2$, with A_{ij} real); (f)F; (g)T; (h)T (it's $\mathrm{id}_3 + e_2(e_3)^{\mathrm{t}}$); (i)T ($y = ((2y - 3x) + 3x)/2$).