Answers to problems in chapter 3

3.1 since every column is either bound or free, sufficient to give just the bound columns or the free columns. (a) no bound columns; (b) no free columns; (c) `bound` $= (1,3)$; (d) subtract 2 times second row from first to find that `bound` $= (1,3)$; ($\begin{bmatrix} 0 & 0 & 9 & 2 \\ 1 & 1 & -2 & 2 \end{bmatrix}$, hence row 2 is pivot row for x_1, row 1 is pivot row for x_3, and that uses up all the rows). (e) `bound` $= (1,2,3)$; (row 3 is pivot row for x_1, row 1 is pivot row for x_2, row 2 is pivot row for x_3, and that uses up all the rows). (f) column 1 is the only bound one since it is not 0 and all columns are scalar multiples of it.

3.2 (a) no: $\begin{bmatrix} 1 & -2 & \pi \\ -1 & 2 & 1-\pi \end{bmatrix} \to \begin{bmatrix} 1 & -2 & \pi \\ 0 & 0 & 1 \end{bmatrix}$ (b) yes (for any y since A is invertible, by inspection); the work: $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & -4 & 1 \\ 3 & 4 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$, i.e., no pivot row left for y.

3.3 If $x \in \operatorname{null} A$, then $B(i,:) * x = 0$ for all rows of B except, perhaps, for the row from which we subtracted a multiple from some (other) row. If this is the ith row, then $B(i,:) * x = (A(i,:) - \alpha A(j,:)) * x = A(i,:) * x - \alpha A(j,:) * x = 0 - 0 = 0$. Hence, altogether, $Bx = 0$, i.e., $x \operatorname{null} B$.

The fact that, here, $j \neq i$ becomes important only now since it implies that $B(j,:) = A(j,:)$, hence we can convert the ith row of B back to $A(i,:)$ simply by subtracting $-\alpha$ times row j of B from row i of B. Hence, A is indeed obtainable by exactly the same kind of process that produced B from A, therefore, by the previous paragraph, also $\operatorname{null} A \supset \operatorname{null} B$.

3.4 By (3.4)Observation, the free and bound unknowns for $A? = 0$ are completely determined by $\operatorname{null} A$, while M being 1-1 implies that $\operatorname{null}(MA) = \operatorname{null} A$.

3.5 (a) []; (b) id_n; (c) $[e_1, 0, e_2, 0] \in \mathbb{R}^{2\times 4}$; (d) $\begin{bmatrix} 1 & 1 & 0 & 22/9 \\ 0 & 0 & 1 & 2/9 \end{bmatrix}$; (e) $\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$; (f) $[x]^{\mathrm{t}}$.

3.6 $A \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, hence $\operatorname{rrref}(A) = [1 \quad 1]$, `f` $= (2)$, `b` $= (1)$, so $C(2,:) = (1)$, $C(1,:) = (-1)$, i.e., $C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, hence $\operatorname{null} A = \operatorname{ran} C = \mathbb{R}y$, with $y = (-1,1)$. I.e., $\operatorname{null} A$ is the straight line through 0 and $(-1,1)$. Since $x := (1,0)$ is a particular solution, the general solution is $x + \operatorname{null} A$, the straight line through x and parallel to $\operatorname{null} A$.

3.7 (a) id_n; (b) []; (c) $[e_2, e_4] \in \mathbb{R}^{4\times 2}$; (d) $\begin{bmatrix} -1 & -22/9 \\ 1 & 0 \\ 0 & -2/9 \\ 0 & 1 \end{bmatrix}$; (e) $\begin{bmatrix} -11 \\ -1/2 \\ -3 \\ 1 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3.8 $MA($: `bound`$) = (MA)(:,$`bound`$) = R(:,$`bound`$) =$ id.

3.9 (a) $[] \in \mathbb{R}^{m\times 0}$; (b) $A = \mathrm{id}_n$; (c) $A(:,[13]) = [e_1, e_2] \in \mathbb{R}^{2\times 2}$; (d) $A(:,[13]) = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$; (e) $A(:,1{:}3) = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & -3 \end{bmatrix}$; (f) $[x]$.

3.10 (a) A; (b) x_1 is the only free unknown; (c)$C = [e_1] \in \mathbb{R}^{n\times 1}$; $A(:,2{:}n)$.

3.11 (a) id_3; (b) $\operatorname{null} M = \operatorname{null} \operatorname{rref}(M) = \operatorname{null} \mathrm{id}_3 = \{0\}$; (c) $L := [e_3, e_2, e_1, 0, 0, 0] \in \mathbb{R}^{3\times 6}$; (d) need to map $e_1, e_2, e_3 \in \mathbb{R}^6$ to 0; so, $P := [e_4, e_5, e_6]^{\mathrm{t}}$ does the job, since $x \in \operatorname{null} P$ if and only if $x_4 = x_5 = x_6 = 0$, i.e., iff $x \in \operatorname{ran}[e_1, e_2, e_3] = \operatorname{ran} M$.

3.12 (a) $[e_1, e_2, e_3, 0, 0, 0] \in \mathbb{R}^{3\times 6}$; (b) $N(:,$`bound`$) =$ id, hence $\operatorname{ran} N = \operatorname{tar} N$; (c) since $N = M^{\mathrm{t}}$, get from previous (c) that $NL^{\mathrm{t}} = (LM)^{\mathrm{t}} = \mathrm{id}^{\mathrm{t}} = \mathrm{id}$.

3.13 Elimination applied to $[U, V]$ shows first two columns bound, the rest free, hence in $\operatorname{ran} U(:,1{:}2)$, so $\operatorname{ran} V \subset \operatorname{ran} U(:,1{:}2) \subset \operatorname{ran} U$. Elimination applied to $[V, U]$ shows the first two columns bound, the rest free, hence $\operatorname{ran} U \subset \operatorname{ran} V$.

If you know about the dimension of a vector space, then already the first calculation shows that $1 \le \dim \operatorname{ran} V \le \dim \operatorname{ran} U = 2$, hence the second elimination calculation need only be done on V, to verify that V is 1-1, and so conclude that $\operatorname{ran} V = \operatorname{ran} U$.

3.14 $A := \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -5 & -2 \\ 1 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -4.5 \\ 1 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ shows that all unknowns are bound, hence the matrix is 1-1, and all equations are used as pivot equation, hence the last unknown in $[A, y]$ is free for any choice of y, hence A is also onto.

3.15 Elimination: $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & k & 6 & 6 \\ -1 & 3 & k-3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & k+4 & 0 & 4 \\ 0 & 1 & k & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & k(k+4) & k \\ 0 & 1 & k & 1 \end{bmatrix}$.

(c) First and second columns are bound. The third column is bound iff $k(k+4) \neq 0$, in which case the last column is free, hence $A? = y$ has exactly one solution then.

(a) Otherwise, when $k = 0$, then the second row is trivial, and the last two columns are free, hence $[A, y]? = 0$ has infinitely many solutions.

(b) Otherwise, if $k = 4$, then the third column is free but the last column is bound, hence there's no solution at all.

3.16 (i) If both were right, then $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ would have a left inverse and a right inverse, yet these would be different, and that can't be.

(ii) We are given a left inverse V for the *square* matrix $A := \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, hence know that $V = A^{-1}$. Writing e_1 as a weighted sum of $(1, 3), (2, 5)$ is precisely the task of writing e_1 as Aa for some a, which is the same as $A^{-1}e_1 = a$, therefore a is the first column of V, i.e., $(-5, 3)$.

(iii) The second factor is 2-by-3, hence must have a nontrivial nullspace, hence the product must have a nontrivial nullspace, hence cannot be the identity.

3.17 (a), (b) not square, hence not invertible;

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}$; elimination of second unknown will zero out the third row, hence third column free, hence not invertible. (d) Matrix differs from (c) only in (3,3) entry, hence now all unknowns are bound and, since matrix is square, it is invertible. To compute: $\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 4 & 0 & 0 & 1 \end{bmatrix} \to$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -4 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & 4 & -5 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 & 4 & -1 \\ 0 & 1 & 0 & 4 & -5 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{bmatrix}$$

and the last three columns form A^{-1}.

(e) $A^{-1} = \begin{bmatrix} 4 & -5 & 2 \\ -4 & 7 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

(f) This matrix is square, hence if it has a left inverse, then that must be its inverse. But, seeing what the matrix does to the unit vectors, it's easy to give a matrix that undoes the action, namely: leave e_2 and e_4 unchanged, subtract e_4 from e_3, and add the result to e_1. In other words, $A^{-1} = [e_1 + e_3 - e_4, e_2, e_3 - e_4, e_4]$.

3.18 (a) $Q : f \mapsto (f(0), f(1), f(2))$ and $V = [()^0, ()^2, ()^4]$ and want Va so that $QVa = \Lambda^{t}()^1 = (0, 1, 2)$. Hence, augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 16 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 12 & -2 \end{bmatrix}$, hence $a_3 = -1/6$, $a_2 = 1 - (-1/6) = 7/6$, $a_1 = 0$. So, desired element is $p = (7()^2 - ()^4)/6 = (7 - ()^2)()^2/6$. Check it: $p(0) = 0$, $p(1) = 6/6 = 1$, $p(2) = (3) * 4/6 = 2$.

(b) Change to $Q : f \mapsto (f(0), f(1), f(-1))$ changes augmented matrix to
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, hence third column free, last column bound, hence no solution.

Could have predicted that since all columns of V are even functions, hence satisfy $v(-1) = v(1)$, while the function $f = ()^1$ we are trying to match is odd, hence satisfies $f(-1) = -f(1)$.

3.19 (a)T; (b)T; (c)F; (d)T; (e)F; (f)F; (g)T; (h)F; (i)T; (j)T.