4.1 A maps e_j to e_{j-1} for all j, with $e_0 := 0$, hence A^n maps all e_j to 0, i.e., $A^n = [A^n e_1, \ldots, A^n e_n] = [0, \ldots, 0] = 0$.

4.2 Certainly, for any $\alpha \in \mathbb{F}$, $A, B \in L(X)$, $\alpha(AB) = A(\alpha B)$, while $\mathrm{id} A = A\,\mathrm{id}$, hence $\alpha\,\mathrm{id}$ commutes with every $B \in L(X)$.

Now assume that A commutes with every $B \in L(X)$. Let $V = [v_1, \ldots, v_n]$ be a basis for X, with inverse $\Lambda^{\mathrm{t}} = [\lambda_1, \ldots, \lambda_n]^{\mathrm{t}}$. Then, for all i, j, $[Av_j] = A[v_j](\lambda_i v_i) = A([v_j]\lambda_i)v_i = ([v_j]\lambda_i)Av_i = [v_j](\lambda_i A v_i)$. In particular, $Av_j = \alpha_j v_j$ for some α_j, therefore $\lambda_i A v_i = \alpha_i$, and so, altogether, $\alpha_j = \alpha_i$. So, $AV = \alpha V$ (with $\alpha := \alpha_j$), and since V is a basis, this implies by (4.2) that $A = \alpha\,\mathrm{id}_X$.

4.3 (i) For each $y \in \mathbb{F}^m$, $x \mapsto y^{\mathrm{t}} A x$ is the map $(y^{\mathrm{t}} A)^{\mathrm{t}}$, hence linear; for each $x \in X$, $y \mapsto y^{\mathrm{t}} A x$ is the map $(Ax)^{\mathrm{t}}$, hence linear.

(ii) Let $A \in \mathbb{F}^{m \times n}$ be given by $A(i,j) := f(e_i, e_j)$. Then, using bilinearity, $f(y,x) = f(\sum_i y_i e_i, \sum_j x_j e_j) = f(y,x) = \sum_i \sum_j y_i x_j f(e_i, e_j) = y^{\mathrm{t}} A x$, i.e., $f = f_A$. If also $f = f_B$, then, in particular, $B(i,j) = f(e_i, e_j) = A(i,j)$ for all i, j, hence $B = A$.

(iii) By (i) and (ii), $\mathbb{F}^{m \times n} \to BL(\mathbb{F}^m, \mathbb{F}^n) : A \mapsto f_A$ is well-defined, 1-1 and onto, hence invertible. Also, $y^{\mathrm{t}}(\alpha A + B)x = \alpha y^{\mathrm{t}} A x + y^{\mathrm{t}} B x$ holds for all $A, B \in \mathbb{F}^{m \times n}$, $\alpha \in \mathbb{F}$, $(x,y) \in \mathbb{F}^m \times \mathbb{F}^n$, showing $A \mapsto f_A$ to be linear.

4.4 (a) try things like `n=7; ab = rand(1,2); xy = rand(3,n); xx = linspace(0,1,101);`

```
max(abs(ab*[interp1(xy(1,:),xy(2,:),xx,'spline');
interp1(xy(1,:),xy(3,:),xx,'spline')]- interp1(xy(1,:),ab*xy(2:3,:),xx,'spline')))
```

which should print out zero (except, perhaps, for round-off).

(b) Let V denote the map, and let $\Lambda^{\mathrm{t}} : f \mapsto f(\mathrm{x})$. Then the description assures us that $\Lambda^{\mathrm{t}} V = \mathrm{id}$, hence V must be 1-1.

(c) `x = 0:3; yy = eye(4); xx = linspace(0,3,121);c = 'r','k','y','c'; for j=1:4; plot(xx,interp1(x,yy`
`hold on, end hold off`

(d) the word 'spline' suggests that these functions might be piecewise cubic. Hence computing third differences (via `diff(vals,3)`) of function values `vals` = f(`xx`) at equally spaced points `xx` should produce a piecewise constant sequence; etc.

4.5 Apply (3.2) to W; the bound columns constitute a basis for $\operatorname{ran} W$.

4.6 $$\begin{bmatrix} 0 & 2 & 0 & 2 & 5 & 4 & 0 & 6 \\ 0 & 1 & 0 & 1 & 2 & 2 & 0 & 3 \\ 0 & 2 & 0 & 2 & 5 & 4 & -1 & 7 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Hence `bound` $= (2,5,7)$, i.e., $V = W(:, [2,5,7]) = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 2 & 0 \\ 2 & 5 & -1 \end{bmatrix}$.

4.7 By definition, $X := \Pi_2(\mathbb{R}^2) = \operatorname{ran} V$ with $V := [()^\alpha : |\alpha| < 3] = [()^{0,0}, ()^{1,0}, ()^{0,1}, ()^{2,0}, ()^{1,1}, ()^{0,2}]$ having 6 columns. Hence sufficient to show that V is 1-1. For the 'data map' $\Lambda^{\mathrm{t}} : p \mapsto (p(0), D_1 p(0), D_2 p(0), D_1^2 p(0), D_1 D_2 p(0),$ we get $\Lambda^{\mathrm{t}} V = \operatorname{diag}(1,1,1,2,1,2)$, an invertible matrix. Hence V is 1-1.

4.8 Being of dimension n, the vector space has a basis $V = [v_1, \ldots, v_n]$. For each $j = 0{:}n$, $[v_1, \ldots, v_j]$ is 1-1, hence a basis for its range, hence that range is a subspace of dimension j.

4.9 (a),(b) Since $DI = \mathrm{id}$, D is onto, hence $\operatorname{ran} D = \operatorname{tar} D = \Pi_{k-1}$. Dimension Formula gives that $\dim \operatorname{null} D = (k+1) - k = 1$, and $D()^0 = 0$, hence $\operatorname{null} D = \operatorname{ran}[()^0]$. Also, I is 1-1, i.e., $\ker I = \{0\}$, hence Dimension Formula gives that $\dim \operatorname{ran} I = \dim \operatorname{dom} I = k$. Also, $()^j \in \operatorname{ran} I$ for $j > 0$, and $[()^j : j > 0]$ is 1-1 (since it is contained in a basis) and has $k = \dim \operatorname{ran} I$ columns, therefore $\operatorname{ran} I = \operatorname{ran}[()^j : j > 0]$.

(c) $Ap = 0$ implies $Dp = -p$ hence $D^{k+1} p = (-1)^{k+1} p$ while, by (a), $D^{k+1} p = 0$ for any $p \in \Pi_k$. So, $\operatorname{null} A = \{0\}$, hence A is 1-1, therefore, by Dimension Formula, A must be onto, i.e., $\operatorname{ran} A = \Pi_k$.

4.10 $V = V_4 A$, with $V_4 := [()^0, \ldots, ()^4]$ 1-1, and $A := \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$ upper triangular with nonzero diagonal entries, hence invertible by (3.19). Hence V is 1-1, with $5 = \dim \operatorname{tar} V$ columns, hence a basis for its target.

4.11 By (4.13), $0 \le \dim Y_0 < \dim Y_1 < \cdots < \dim Y_r \le \dim X$.

4.12 If $\dim Y = k$, then $Y = \operatorname{ran}[v_1] \dot{+} \cdots \dot{+} \operatorname{ran}[v_k]$ for every basis $[v_1, \ldots, y_k]$ for Y and, by (i) and induction, $d(Y) = \sum_j \dim \operatorname{ran}[v_j]$, and this equals k by (ii). Hence, for any Y, $d(Y) = \dim Y$.

4.13 Let $B = A|_Z$; then $A(Z) = \operatorname{ran} B$ and $\operatorname{null} B = Z \cap \operatorname{null} A$, hence the Dimension Formula finishes the proof.

4.14 (a) Let V be a basis for $\operatorname{null} B$; since $\operatorname{null} B \subset \operatorname{null}(AB)$, we may extend V to a basis $[V, W]$ for $\operatorname{null}(AB)$. Then BW is a 1-1 column map into $\operatorname{null} A$, hence $\#W \le \operatorname{defect}(A)$ and therefore $\operatorname{defect}(AB) = \#V + \#W \le \operatorname{defect}(B) + \operatorname{defect}(A)$.

(b) Since $\operatorname{ran} AB \subset \operatorname{ran} A$, have $\operatorname{defect}(A) = \dim \operatorname{dom} A - \dim \operatorname{ran} A \le \dim \operatorname{dom} A - \dim \operatorname{ran} AB = \dim \operatorname{dom} AB - \dim \operatorname{ran} AB = \operatorname{defect}(AB)$, using the Dimension Formula twice.

(c) By (b), we must make certain that $\dim \operatorname{dom} B \ne \dim \operatorname{dom} A$. However, for AB to be defined, we must have $\operatorname{ran} B \subset \operatorname{dom} A$. So, how about $B = [] : \mathbb{R}^0 \to \mathbb{R}^1$ and $A : \mathbb{R}^1 \to \mathbb{R}^1 : x \mapsto 0$?

4.15 If $A = CB$, then $Bx = 0$ implies $Ax = CBx = 0$, hence $\operatorname{null} B \subset \operatorname{null} A$.

For the converse, let $[U, V]$ be a basis for $X = \operatorname{dom} A = \operatorname{dom} B$ so that U is a basis for $\operatorname{null} B$. Then BV is 1-1, hence extendible to a basis $[BV, W]$ of Z. By (4.2), there is exactly one linear map $C : Z \to Y$ with $CBV = AV$, $CW = 0$. But then, $CB[U, V] = [0, CBV] = [0, AV] = [AU, AV] = A[U, V]$, showing A and CB to agree on a basis for their common domain, and this implies that they are equal.

4.16 $\dim \operatorname{ran}(AB) = \dim \operatorname{ran} B - \dim(\operatorname{null} A \cap \operatorname{ran} B)$, while $\dim \operatorname{ran}(BC) = \dim \operatorname{ran}(ABC) + \dim(\operatorname{null} A \cap \operatorname{ran}(BC))$, and $\operatorname{ran}(BC) \subset \operatorname{ran} B$, hence also $\dim(\operatorname{null} A \cap \operatorname{ran}(BC)) \le \dim(\operatorname{null} A \cap \operatorname{ran} B)$.

4.17 Since Y is a linear subspace, we have $(x + Y) + (z + Y) = (x + z) + Y$ and, for any nonzero α, $\alpha(x + Y) = (\alpha x) + Y$, while $0(x + Y) = Y$, by definition. Hence the map f is linear. Also, $\operatorname{null} f = \{x \in X : x + Y = Y\} = Y - Y = Y$.

(ii) By (i), X/Y is the range of the map f that satisfies (2.5), and (ii) follows from Problem 2.7.

(iii) By (i) and the Dimension Formula, $\dim X/Y = \dim \operatorname{ran} f = \dim \operatorname{dom} f - \dim \operatorname{null} f = \dim X - \dim Y = \operatorname{codim} Y$.

4.18 Let $T := \{(i, j) \in \underline{n} \times \underline{n} : i \le j\}$. Then the map $\mathbb{F}^{n \times n} \to \mathbb{F}^T : A \mapsto A|_T : (i, j) \mapsto A_{ij}$ is linear, and maps the subspace of all upper triangular matrices of order n 1-1 onto $\mathbb{F}^T$, hence that subspace has dimension $\dim \mathbb{F}^T = \#T = (n + 1)n/2$.

4.19 (i) By (4.7), any basis V for Y can be extended to a basis $[V, W]$ for X, and, by (4.26), $X = Y \dot{+} \operatorname{ran} W$, i.e., $\operatorname{ran} W$ is a complement for Y.

(ii) If V is a basis for Y and W is a basis for Z, then, by (4.26), Z is a complement for Y iff $[V, W]$ is a basis. In particular, $\dim Z = \#W = \dim X - \dim Y$, a number that depends on X and Y alone.

(iii) From (ii), $\operatorname{codim} Y = \dim X - \dim Y$, while, by the Dimension Formula, $\dim(Y + Z) = \dim Y + \dim Z - \dim(Y \cap Z)$, so, on subtracting this equation from the identity $\dim X = \dim X + \dim X - \dim X$, the proof is finished.

(iv) If $X = Y \dot{+} Z$ with both Y and Z proper, then there is a basis $[V, W]$ for X, with V a basis for Y and W a basis for Z. If W is neither X nor $\{0\}$, then $V = [v_1, \ldots, v_r]$ for some $r > 0$ and $W = [w_1, \ldots, w_s]$ for some $s > 0$, and $w_1 + v_1 \notin Z$, therefore $Z_1 := \operatorname{ran}[w_1 + v_1, w_2, \ldots] \ne Z$, yet $[V, w_1 + v_1, w_2, \ldots, w_s]$ is 1-1 hence a basis for X and therefore also Z_1 is a complement for Y different from Z.

(v) By (ii), if $\operatorname{codim} Y > \dim Z$, then $\dim X > \dim Y + \dim Z \ge \dim(Y + Z)$.

(vi) By (ii), if $\dim Y > \operatorname{codim} Z$, then $\dim Y + \dim Z > \dim X$, hence, by Dimension Formula, $\dim(Y \cap Z) > 0$.

4.20 If $\sum_j d_j = n$, then we can partition the n columns of any basis V for X into $V = [V_1, \ldots, V_r]$ with $\#V_j = d_j$, and conclude from (4.26) that then $X = Y_1 \dotplus \cdots \dotplus Y_r$, with $Y_j := \operatorname{ran} V_j$ of dimension $\#V_j = d_j$, all j.

Conversely, if we have such a direct sum decomposition for X, then, by (4.26), $[V_1, \ldots, V_r]$ is a basis for X, hence $\sum_j d_j = \dim X = n$.

4.21 If $\dim Y = k$, then $Y = \operatorname{ran}[v_1] \dotplus \cdots \dotplus \operatorname{ran}[v_k]$ for every basis $[v_1, \ldots, y_k]$ for Y and, by (i) and induction, $d(Y) = \sum_j \dim \operatorname{ran}[v_j]$, and this equals k by (ii). Hence, for any Y, $d(Y) = \dim Y$.

4.22 Straightforward verification. E.g., $\alpha((y_1, \ldots, y_r) + (z_1, \ldots, z_r)) = \alpha(y_1 + z_1, \ldots, y_r + z_r) = (\alpha(y_1 + z_1), \ldots, \alpha(y_r + z_r)) = (\alpha y_1 + \alpha z_1, \ldots, \alpha y_r + \alpha z_r) = (\alpha y_1, \ldots, \alpha y_r) + (\alpha z_1, \ldots, \alpha z_r) = \alpha(y_1, \ldots, y_r) + \alpha(z_1, \ldots, z_r)$ verifies (s.3) of (2.1).

4.23 Recall the basis $V_k := [()^0, \ldots, ()^k]$ for Π_k.

(a) $V = V_3 A$, with $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 1-1 by inspection, hence V is 1-1, giving $\dim \operatorname{ran} V = \dim \operatorname{dom} V = 3 < 5 = \dim \operatorname{tar} V$, i.e., V is not onto.

(b) $V = V_4 A$, with $A = \begin{bmatrix} 0 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 1-1 by inspection (e.g., thinking of elimination from right to left, row j is pivot row for column $6 - j$, all j), hence V is 1-1, $\dim \operatorname{ran} V = \dim \operatorname{dom} V = 4 < 5 = \dim \operatorname{tar} V$, i.e., V is not onto.

(c) $V = V_4 A$, with $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$. Using the rows from last to second to eliminate unknowns, we obtain the equivalent matrix $A \to B = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$, and now we see that all columns of A are bound, hence A is 1-1, therefore $\dim \operatorname{ran} V = \dim \operatorname{dom} V = 5 = \dim \operatorname{tar} V$, i.e., V is also onto.

4.24 (a) V maps into Π_2, a 3-dimensional space, hence at most 3 columns can be bound. Since $V = [()^2, ()^1, ()^0] \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 0 & -2 & -4 & \cdots \\ 0 & 1 & 4 & \cdots \end{bmatrix}$, and using the last row of this matrix as pivot row for the second column gives the modified second row $[\,0 \quad 0 \quad 4 \quad \cdots]$, this shows that the first three columns of V are bound, hence the others must be free. Therefore $[f_0, f_1, f_2]$ is a basis for $\operatorname{ran} V$, and that smallest n is 3.

Alternative argument: since $\operatorname{ran} V \subset \Pi_2$, have $\dim \operatorname{ran} V \le \#[()^2, ()^1, ()^0] = 3$, while, with $t = 0{:}2$, have $[f_0(t), f_1(t), f_2(t)] = [\,t^2, (t-1)^2, (t-2)^2\,] = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 8 \\ 1 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix}$, showing all three columns to be bound, hence $[f_0, f_1, f_2]$ must be 1-1, therefore a basis for $\operatorname{ran} V$, hence the other columns of V must be free. In particular, $n = 2$.

(b) $V = [\sin, \cos] \begin{bmatrix} 1 & \cos(1) & \cdots \\ 0 & \sin(1) & \cdots \end{bmatrix}$, hence first two columns of matrix are bound, the others must be free. Since $[\sin, \cos]$ is 1-1 (e.g., $[\delta_{\pi/2} \delta_0]^{\mathrm{t}} [\sin, \cos] = \mathrm{id}_2$) also the first 2 columns of V are bound and the rest are free.

(c) $V = [\exp][1, \ldots]$, hence the first column of V is bound and the others are free, i.e., $[\exp]$ is a basis for $\operatorname{ran} V$. So, $n = 1$.

4.25 Each w_j is the product of k linear factors, hence of exact degree k, therefore W maps into Π_k. Also, $\#W = k + 1 = \dim \Pi_k$. Hence, W is a basis for Π_k if and only if W is 1-1.

The matrix $QW = (w_j(\tau_{k+1+i}) : i, j = 0{:}k)$ is upper triangular with nonzero diagonal entries, hence invertible by (3.19)Proposition, therefore W must be 1-1.

4.26 Each w_j is the product of k linear factors, hence of exact degree k, hence W maps into Π_k. Also, $\#W = k + 1 = \dim \Pi_k$. Hence, W is a basis for Π_k if and only if W is 1-1.

Let $Q : p \mapsto (D^{n_j} p(\tau_j) : j = 1{:}k + 1)$, with $n_j := \max\{i \le k + 1 : \tau_i = \tau_j\} - j$, all j. Then $QW =$ $\begin{bmatrix} 0 & \times & & & \\ 0 & 0 & \times & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \times \\ \times & 0 & 0 & \cdots & 0 \end{bmatrix}$, with $\times$ indicating elements that are guaranteed to be nonzero in case $\tau_k < \tau_{k+1}$ since each is the value of the n_jth derivative at τ_j of a polynomial that has a root of exact order n_j at τ_j. Hence, in that case every column of the square matrix QW is bound. This makes QW invertible, hence W is 1-1. If, on the other hand, $\tau_k = \tau_{k+1}$, then $\times$ in the lower left corner becomes zero since it is $w_0(\tau_{k+1})$, hence then $w_j(\tau_{k+1}) = 0$ for every $j = 0{:}k$, therefore every $p \in \operatorname{ran} W$ vanishes at τ_{k+1}, hence $\operatorname{ran} W$ cannot be Π_k.

4.27 (a)T; (b)T; (c)F; (d)F.