Answers to all problems in chapter 5

5.1 (a) no; (b) yes (it’s [3, 0, −2]); (c) no (neither additive nor homogeneous); (d) yes; (e) yes; (f) no.

5.2 (a) \[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix},
\]
so \( U = [1, 0] \) will do. (b) Read off directly that \( U = V \). (c) \[
R(t:2,1:2) = \text{id}, \text{ hence } U = \begin{bmatrix}
1 & 0 & -1/3 \\
0 & 0 & 1/6 \\
0 & 1 & 0
\end{bmatrix}
\text{ will do. (d) } \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & -2 & 0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & -2 & 0 & 0 & 0 & 1
\end{bmatrix} = R.
\]
R. At this point, \( R([1,3],[1,2]) = \text{id} \), hence \( U := R([1,3],[3:6]) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix} \) will do, as is verified by computing the product \( UV \) which is \text{id} (as it should).

5.3 Since \([V, \text{id}_n]\) is onto, it must have exactly \( n \) bound columns, hence \#b = n − r, therefore \#f = r, hence \( V(f,.;.) \) is square.

5.4 \((p(r_1), \ldots, p(r_k))\)

5.5 We know that \( \Lambda^t p := (p(1), Dp(1), p(-1)) = (3, 6, 3) \) and that \( p \in \Pi_2 = \text{ran} V \), with \( V := \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 3 \\
1 & 1 & 3
\end{bmatrix} \) as solution to \( \Lambda^t V \), which is \text{id}.

5.6 Since \( p(0) \) is the coefficient of \( (0)^0 \) when writing \( p \) in terms of \( V = \begin{bmatrix}(0)^0, (0)^1, (0)^2\end{bmatrix} \), we are looking for the first row of \( (\Lambda^t V)^{-1} \), with \( \Lambda^t p = (p(1), Dp(1), p(-1)) \). The first row of the \text{rrref}([\Lambda^t V, \text{id}_3]) is \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 \\
0 & 1 & 0 & 0
\end{bmatrix}, \text{ hence } p(0) = a_1 = 0.
\]
Alternatively, from the answer to next problem, \( p(0) = (3 * 3 - 2 * 6 + 3)/4 = 0 \).

5.7 \( V^{-1} W = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} =: A \), hence can find coordinates wrto \( W \) as solution to \( A? = (3, -4, 2): \)

\[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ i.e., } q = W(7, -6, 2).
\]

5.8 Since \( \Lambda^t \big|_{\chi} \) is invertible, it is sufficient to check that, e.g., \( W \) is a right inverse. But that is obvious: \( \Lambda^t W = \Lambda^t V (\Lambda^t V)^{-1} = \text{id} \).

5.9 \( V^{-1} W = \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} =: A \), hence can find coordinates wrto \( W \) as solution to \( A? = (3, -4, 2): \)

\[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \text{ i.e., } q = W(7, -6, 2).
\]

5.10 With \( \Lambda^t \) as defined, \( W(v_1, \ldots, v_n; x) = \Lambda^t V \), hence its invertibility implies that \( V \) must be 1-1.

5.11 \( F^2 = \text{id} \) and \( Q = (\text{id} + F)/2 \), hence \( F Q = Q \) showing that \( \text{ran} Q \subset f : F f = f \subset f : Q f = f \subset \text{ran} Q \), hence \( Q \) is a linear projector with range the \text{even} functions, meaning the functions \( f \) for which \( f(-t) = f(t) \) for all \( t \). Also, \( Q \) consists of the \text{odd} functions, meaning the functions for which \( f(-t) = -f(t) \) for all \( t \).