Answers to all problems in chapter 5

5.1 (a) no; (b) yes (it's $[3, 0, -2]$); (c) no (neither additive nor homogeneous); (d) yes; (e) yes; (f) no.

5.2 (a) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, so $U = [1, 0]$ will do. (b) Read off directly that $U = V$. (c) $\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 1/6 \\ 0 & 0 & -2 & 1 & 0 \end{bmatrix} =: R$, so $R(1{:}2, 1{:}2) = \mathrm{id}$, hence $U = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 1/6 \end{bmatrix}$ will do. (d) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} =: R$. At this point, $R([1, 3], [1, 2]) = \mathrm{id}$, hence $U := R([1, 3], 3{:}6) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ will do, as is verified by computing the product UV which is id_2 (as it should).

5.3 Since $[V, \mathrm{id}_n]$ is onto, it must have exactly n bound columns, hence $\#\mathtt{b} = n - r$, therefore $\#\mathtt{f} = r$, hence $V(\mathtt{f}, :)$ is square.

5.4 $(p(\tau_1), \ldots, p(\tau_k))$

5.5 We know that $\Lambda^{\mathrm{t}} p := (p(1), Dp(1), p(-1)) = (3, 6, 3)$ and that $p \in \Pi_2 = \operatorname{ran} V$, with $V := [()^0, ()^1, ()^2]$, i.e., $p = Va$, while $\Lambda^{\mathrm{t}} V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, so $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 6 \\ 1 & -1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, hence $p(0) = a_1 = 0$.

Alternatively, from the answer to next problem, $p(0) = (3 * 3 - 2 * 6 + 3)/4 = 0$.

5.6 Since $p(0)$ is the coefficient of $()^0$ when writing p in terms of $V = [()^0, ()^1, ()^2]$, we are looking for the first row of $(\Lambda^{\mathrm{t}} V)^{-1}$,with $\Lambda^{\mathrm{t}} p = (p(1), Dp(1), p(-1))$. The first row of the rref$([\Lambda^{\mathrm{t}} V, \mathrm{id}_3])$ is $[1 \quad 0 \quad 0 \quad 3/4 \quad -1/2 \quad 1/4]$, hence $p(0) = (3p(1) - 2Dp(1) + p(-1))/4$.

5.7 $V^{-1} W = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =: A$, hence can find coordinates wrto W as solution to $A? = (3, -4, 2)$: $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, i.e., $q = W(7, -6, 2)$.

5.8 Since $\Lambda^{\mathrm{t}}|_X$ is invertible, it is sufficient to check that, e.g., W is a right inverse. But that is obvious: $\Lambda^{\mathrm{t}} W = \Lambda^{\mathrm{t}} V (\Lambda^{\mathrm{t}} V)^{-1} = \mathrm{id}$.

5.9 $V^{-1} W = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =: A$, hence can find coordinates wrto W as solution to $A? = (3, -4, 2)$: $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, i.e., $q = W(7, -6, 2)$.

5.10 With Λ^{t} as defined, $W(v_1, \ldots, v_n; x) = \Lambda^{\mathrm{t}} V$, hence its invertibility implies that V must be 1-1.

5.11 $F^2 = \mathrm{id}$ and $Q = (\mathrm{id} + F)/2$, hence $FQ = Q$ showing that $\operatorname{ran} Q \subset f : Ff = f \subset f : Qf = f \subset \operatorname{ran} Q$, hence Q is a linear projector with range the *even* functions, meaning the functions f for which $f(-t) = f(t)$ for all t. Also, Q consists of the *odd* functions, meaning the functions for which $f(-t) = -f(t)$ for all t.