Answers to all problems in chapter 6

6.1 \((1, -1, 1)^T(1, 1, 1)/(1, -1, 1)^T(1, 1, 1) = 1/3\), hence the projection is \((1, 1, 1)/3\).

6.2 \(\alpha = \alpha^v(x - y)/(\alpha^v v) = -2/3\), hence the projection is \(y + \alpha v = (7, -2, 5)/3\).

6.3 To minimize \(\|y - z - [v, w][(-\alpha, \beta)]\| \) over all \((-\alpha, \beta) \in \mathbb{R}^2\), let \(V := [v, w]\). Then \(V^c V, V^c(y - z) = 3.2, 4 \rightarrow 0, -1, 5/2, 2, 0, 0, 6\), hence \((-\alpha, \beta) = (3, -5/2)\), therefore the distance is \(\|(0, -3/2, 3/2)\| = \sqrt{2(3/2)} = 2.121 \cdots\).

6.4 With \(V := [v_1, v_2], V^c V = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}\), hence \(P_V = VV^c/9 = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}/9\).

(b) \(P_V y = (7, 8, 11)/9\).

6.5 (a) \(V^c V = \begin{bmatrix} f(0) & f(1) \\ f(1) & f(2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix}\), hence \(P_V = V = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}\).

(b) \(V^c(v^c)^2 = (f(2), f(3) = (2/3, 0), \) therefore \(P_V(v^c)^2 = V(1, 3/0) = 1/3(0)\).

6.6 From proof of (6.11), \(|u + v|^2 = \|u\|^2 + \|v\|^2 + 2\) if and only if \(v^c u + u^c v = 0\). If \(F = \mathbb{R}\), then \(v^c u = u^c v\), hence this happens if and only if \(v^c = 0\), i.e., \(u \perp v\). If \(F = \mathbb{C}\), then \(v^c u = \bar{u}^c v\), hence this happens if and only if \(u^c v^c\) is purely imaginary, which is not the same as saying that \(u \perp v\).

6.7 (a) With \(V = [0, 0, 1]\), and \(t = 1:10\), and \(y = (1, 4, 9, \ldots, 100)\), \(V^c V = \left[ \sum_{j=0}^{10} \sum_{j=0}^{10} \sum_{j=0}^{10} \right] = \left[ \begin{array}{cccc} 10 & 55 & 385 \\ 55 & 385 & 3025 \end{array} \right], \) while \(V^c y = (385, 3025), \) so \(V^c V, V^c y = \left[ \begin{array}{cccc} 10 & 55 & 385 \\ 55 & 385 & 3025 \end{array} \right] \rightarrow \left[ \begin{array}{cccc} 1 & 5.5 & 38.5 \\ 0 & 82.5 & 907.5 \end{array} \right] \rightarrow \left[ \begin{array}{cccc} 1 & 0 & -22 \\ 0 & 1 & 11 \end{array} \right]\)

Hence \(11(0)^1 - 2\) is it. (b) Now \(V^c V = \left[ \begin{array}{cccc} 10 & 0 & 0 \\ 0 & 0 & 82.5 \end{array} \right] \) and \(V^c y = (385, 907.5)\), and 907.5/82.5 = 11, hence the discrete least-squares straight line approximation is 38.5\(0^1 + 11(0)^1 - 5.5 = 11(0)^1 + (38.5 - 11 * 5.5) = 11(0)^1 - 22(0)^0\). (c) In (b), \(V^c V\) is diagonal, hence the normal equations \(V^c V? = V^c y\) are easier to solve.

6.8 (a) No \((f(2, 2), e_2) = 0\); (b) No \((f(1, 1, 0), (1, 1, 0) = 0)\); (c) No \((x \mapsto f(x, y) \) not linear, e.g., not homogeneous); (d) No \((f((1, i, 0), (1, 1, 0)) = 0)\); (e) No \((f((1, -1, 1), (1, -1, 1)) \) is not positive;

6.9 (a) \(\langle x, x \rangle = (Ax)^c Ax = \|Ax\|^2 \geq 0\), with equality iff \(Ax = 0\), i.e., \(x = 0\), since \(A\) is invertible. (b) For any \(y, y^c A^c A\) is a composition of linear maps, hence linear. (c) \((Ax)^c Ay = (Ay)^c Ax\).

6.10 \(V^c V = \text{diag}(2, 3, 6)\) is invertible, and \#V = \(\text{dim} \mathbb{R}^3\). \(V^{-1} x = \text{diag}(1/2, 1/3, 1/6) \ast V^c x = e_2\). (Of course, since \(V(2, 2) = x\), no need to actually calculate the coordinates of \(x\) wto \(V\) :)

6.11 \(V^c V = \text{diag}(3, 6, 11)\) and \((V^c V)^{-1}V^c e_4 = (0, 0, 1/3)\), therefore \(P_V e_4 = (1/3) e_3 = (1, -1, 0, 2)/3 \neq e_4\), thus \([V, e_4]\) with \(e_4 = (2/3, 2/3, 2/3) = (-1, 1, 0, 1)\) is an orthogonal basis for \(\mathbb{R}^4\).

6.12 (a) With \((f, g) := \sum_{j=1}^{10} f(j)g(j), \) we compute \((1)^1, (0)^0)/(1)^0, (0)^0) = 55/10 = 5.5, \) hence \(q_2 := (1)^1 - 5.5 \) is orthogonal to \(q_1 := (0)^0\). Also from H.P. 6.7, \(q_3 \equiv (2 - (11)^1 - 22\) is the error in the discrete least-squares approximation from \(P_1\) to \(1\), hence is orthogonal to \(P_1\). Hence \((0)^0, q_2, q_3\) is an orthogonal basis for \(P_2\).

(b) \((1)^3, q_1)/(1)^0, (0)^0 = 3025/10 = 302.5; (1)^3, q_2)/(q_2, q_2) = 8695.5/82.5 = 105.4; (1)^3, q_3)/(q_3, q_3) = 8712/528 = 16.5. \) Hence the discrete least squares quadratic approximation to \(1\) is \(302.5 + 105.4((1)^1 \cdots 5.5) + 16.5((2 - (11)^1 - 22\)\).

6.13 (a) \(V^c V = \begin{bmatrix} f(0)^0 & f(1)^0 \\ f(1)^0 & f(2)^0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix}\) is diagonal, hence \(V^c V\) is orthogonal.

(b) \(V^c(v^c)^2 = (f(2), f(3) = (2/3, 0), \) hence \(P_V(v^c)^2 = (0(2/3)^2/2 + (1)^0)/(3/2) = (0)^0/3\).

(c) Since \((2 - P_V(v^c)^2 \perp \text{ran} V,\) know that \([0, 1)^1, (2 - (0)^0/3\) is 1-1 and orthogonal, into the 3-dim. space \(P_2\), hence an orthogonal basis for it. \(\|(2 - (0)^0/3)\|^2 = ((2 - (0)^0/3, (2) = 2/5 - 2/9 = 8/45. Normalized: \((0)^0/3, (1)^0/3, (2)^0/3\)\).

6.14 \(1.841 \cdots\) radians.

6.15 Since the first column has length \(\sqrt{k + 1}\), all rows and columns must have that length. In partic-
ular, for each $i$, $\sum_{j=0}^{k} |z_i|^j = k + 1$, hence $|z_i| = 1$. Also, for $i \neq h$, $s := \sum_{j=0}^{k} (\overline{z_h} z_i)^j = 0$, hence $z_h \neq z_i$, and, since $z_h^{-1} = \overline{z_h}$, also $1 - (z_h^{-1} z_i)^{k+1} = (1 - \overline{z_h} z_i)s = 0$, i.e., $z_h = r_{ih} z_i$ with $r_{ih}$ a $(k + 1)$st root of unity. In particular, $z_h = r_h z_1$ with $(r_1, \ldots, r_k)$ pairwise distinct $(k + 1)$st roots of unity. Since there are exactly $k + 1$ such roots, we are done, with $z_1 = \exp(2\pi i \alpha)$.

6.16 (a) F; (b) F (e.g., take $x = y \neq 0$):