Answers to all problems in chapter 6

6.1 $(1,-1,1)^{c}(1,1,1)/(1,-1,1)^{c}(1,-1,1) = 1/3$, hence the projection is $(1,1,1)/3$.

6.2 $\alpha = v^{c}(x-y)/(v^{c}v) = -2/3$, hence the projection is $y + \alpha v = (7,-2,5)/3$.

6.3 To minimize $\|y - z - [v,w](-\alpha,\beta)\|$ over all $(-\alpha,\beta) \in \mathbb{R}^2$, let $V := [v,w]$. Then $[V^{c}V, V^{c}(y-z)] = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 5/2 \\ 2 & 0 & 6 \end{bmatrix}$, hence $(-\alpha,\beta) = (3,-5/2)$, therefore the distance is $\|(0,-3/2,3/2)\| = \sqrt{2}(3/2) = 2.121\cdots$.

6.4 With $V := [v_1, v_2]$, $V^{c}V = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$, hence $P_V = VV^{c}/9 = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}/9$.

(b) $P_V y = (7,8,11)/9$.

6.5 (a) $V^{c}V = \begin{bmatrix} \int ()^0 & \int ()^1 \\ \int ()^1 & \int ()^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix}$, hence $P_V = V \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix} V^{c}$.

(b) $V^{c}()^2 = (\int ()^2, \int ()^3) = (2/3, 0)$, therefore $P_V()^2 = V(1/3,0) = 1/3()^0$.

6.6 From proof of (6.11), $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $v^{c}u + u^{c}v = 0$. If $\mathbb{F} = \mathbb{R}$, then $v^{c}u = u^{c}v$, hence this happens if and only if $u^{c}v = 0$, i.e., $u \perp v$. If $\mathbb{F} = \mathbb{C}$, then $v^{c}u = \overline{u^{c}v}$, hence this happens if and only if $u^{c}v$ is purely imaginary, which is not the same as saying that $u \perp v$.

6.7 (a)With $V = [()^0, ()^1]$, and $t = 1{:}10$, and $y = (1,4,9,\ldots,100)$, ${V_t}^{c}V_t = \begin{bmatrix} \sum ()^0 & \sum ()^1 \\ \sum ()^1 & \sum ()^2 \end{bmatrix} = \begin{bmatrix} 10 & 55 \\ 55 & 385 \end{bmatrix}$, while ${V_t}^{c}y = (385, 3025)$, so $[{V_t}^{c}V_t, {V_t}^{c}y] = \begin{bmatrix} 10 & 55 & 385 \\ 55 & 385 & 3025 \end{bmatrix} \to \begin{bmatrix} 1 & 5.5 & 38.5 \\ 0 & 82.5 & 907.5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -22 \\ 0 & 1 & 11 \end{bmatrix}$. Hence $11(()^1 - 2)$ is it. (b) Now ${V_t}^{c}V_t = \begin{bmatrix} 10 & 0 \\ 0 & 82.5 \end{bmatrix}$ and $V_t y = (385, 907.5)$, and $907.5/82.5 = 11$, hence the discrete least-squares straight line approximation is $38.5()^0 + 11(()^1 - 5.5) = 11()^1 + (38.5 - 11 * 5.5) = 11()^1 - 22()^0$. (c) In (b), ${V_t}^{c}V_t$ is diagonal, hence the normal equations ${V_t}^{c}V_t? = {V_t}^{c}y$ are easier to solve.

6.8 (a) No ($f(e_2,e_2) = 0$); (b) No ($f((1,1,0),(1,1,0) = 0$); (c) No ($x \mapsto f(x,y)$ not linear, e.g., not homogeneous); (d) No ($f((1,\mathrm{i},0),(1,\mathrm{i},0)) = 0$); (e) No ($f((1,-1,1),(1,-1,1))$ is not positive;

6.9 (a) $\langle x,x\rangle = (Ax)^{c}Ax = \|Ax\|^2 \ge 0$, with equality iff $Ax = 0$, i.e., $x = 0$, since A is invertible. (b) For any y, $y^{c}A^{c}A$ is a composition of linear maps, hence linear. (c) $(Ax)^{c}Ay = \overline{(Ay)^{c}Ax}$.

6.10 $V^{c}V = \mathrm{diag}(2,3,6)$ is invertible, and $\#V = \dim \mathbb{R}^3$. So, $V^{-1}x = \mathrm{diag}(1/2,1/3,1/6) * V^{c}x = e_2$. (Of course, since $V(:,2) = x$, no need to actually calculate the coordinates of x wrto V :-)

6.11 $V^{c}V = \mathrm{diag}(3,6,6)$ and $(V^{c}V^{-1}V^{c}e_4 = (0,0,1/3)$, therefore $P_V e_4 = (1/3)v_3 = (1,-1,0,2)/3 \ne e_4$, thus $[V, v_4]$ with $v_4 := 3(e_4 - P_V e_4) = (-1,1,0,1)$ is an orthogonal basis for $\mathbb{R}^4$.

6.12 (a) With $\langle f,g\rangle := \sum_{j=1}^{10} f(j)g(j)$, we compute $\langle ()^1, ()^0\rangle / \langle ()^0, ()^0\rangle = 55/10 = 5.5$, hence $q_2 := ()^1 - 5.5$ is orthogonal to $q_1 := ()^0$. Also from H.P. 6.7, $q_3 := ()^2 - (11()^1 - 22)$ is the error in the discrete least-squares approximation from Π_1 to $()^2$, hence is orthogonal to Π_1. Hence $[()^0, q_2, q_3]$ is an orthogonal basis for Π_2.

(b) $\langle ()^3, q_1\rangle / \langle ()^0, ()^0\rangle = 3025/10 = 302.5$; $\langle ()^3, q_2\rangle / \langle q_2, q_2\rangle = 8695.5/82.5 = 105.4$; $\langle ()^3, q_3\rangle / \langle q_3, q_3\rangle = 8712/528 = 16.5$. Hence the discrete least squares quadratic approximation to $()^3$ is $302.5 + 105.4(()^1 - 5.5) + 16.5(()^2 - 11(()^1 - 2))$.

6.13 (a) $V^{c}V = \begin{bmatrix} \int ()^0 & \int ()^1 \\ \int ()^1 & \int ()^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix}$ is diagonal, hence V is orthogonal.

(b) $V^{c}()^2 = (\int ()^2, \int ()^3) = (2/3, 0)$, hence $P_V()^2 = ()^0(2/3)/2 + ()^1 0/(3/2) = ()^0/3$.

(c) Since $()^2 - P_V()^2 \perp \operatorname{ran} V$, know that $[()^0, ()^1, ()^2 - ()^0/3]$ is 1-1 and orthogonal, into the 3-dim. space Π_2, hence an orthogonal basis for it. $\|()^2 - ()^0/3\|^2 = \langle ()^2 - ()^0/3, ()^2\rangle = 2/5 - 2/9 = 8/45$. Normalized: $[()^0/\sqrt{2}, ()^1\sqrt{3/2}, (()^2 - ()^0/3)\sqrt{45/8}]$.

6.14 $1.841\cdots$ radians.

6.15 Since the first column has length $\sqrt{k+1}$, all rows and columns must have that length. In particular, for each i, $\sum_{j=0}^{k} |z_i|^j = k+1$, hence $|z_i| = 1$. Also, for $i \neq h$, $s := \sum_{j=0}^{k} (\overline{z_h} z_i)^j = 0$, hence $z_h \neq z_i$, and, since $z_h^{-1} = \overline{z_h}$, also $1 - (z_h^{-1} z_i)^{k+1} = (1 - \overline{z_h} z_i)s = 0$, i.e., $z_h = r_{ih} z_i$ with r_{ih} a $(k+1)$st root of unity. In particular, $z_h = r_h z_1$ with $(r_1, \ldots, r_k)$ pairwise distinct $(k+1)$st roots of unity. Since there are exactly $k+1$ such roots, we are done, with $z_1 =: \exp(2\pi \mathrm{i} \alpha)$.

6.16 (a) F; (b) F (e.g., take $x = y \neq 0$);