Answers to all problems in chapter 8

8.1 The rrref for this matrix is

$$\mathrm{rrref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix}$$

hence $A = A(:,[1,2])\mathrm{rrref}(A)$ with $A(:,[1,2])$ a basis for $\mathrm{ran}\, A$, therefore this factorization is minimal.

8.2 From the minimal factorization in previous problem, $A(:,1{:}2)$ is a basis for $\mathrm{ran}\, A$, and $\mathrm{rrref}(A)^{\mathrm{t}}$ is a basis for A^{t}.

8.3 $A = [e_1, e_2, 0, 0]$, $B = [0, 0, e_3, e_4]$

8.4 Let $A = V_A\Lambda_A$ and $B = V_B\lambda_B$ be minimal factorizations, therefore $\mathrm{rank}\, A = \#V_A$, $\mathrm{rank}\, B = \#V_B$. Then $AB = V_A(\Lambda_A V_B \Lambda_B)$ which implies that $\mathrm{rank}(AB) \le \#V_A = \mathrm{rank}\, A$. Also $AB = (V_A\Lambda_A V_B)\Lambda_B$ which implies that $\mathrm{rank}(AB) \le \#(V_A\Lambda_A V_B) = \#V_B = \mathrm{rank}\, B$. Therefore, $\mathrm{rank}(AB) \le \min\{\mathrm{rank}\, A, \mathrm{rank}\, B\}$.

The alternative argument: $\mathrm{ran}\, AB \subseteq \mathrm{ran}\, A$ implies that $\mathrm{rank}(AB) \le \mathrm{rank}\, A$, while also $\mathrm{ran}(AB) = A(\mathrm{ran}\, B) = \mathrm{ran}(A_{|\,\mathrm{ran}\, B})$, hence $\mathrm{rank}\, AB = \dim\mathrm{ran}(AB) = \dim\mathrm{ran}(A_{|\,\mathrm{ran}\, B}) \le \dim\mathrm{dom}(A_{|\,\mathrm{ran}\, B}) = \dim\mathrm{ran}\, B = \mathrm{rank}\, B$, the inequality by the Dimension Formula.

The most direct argument: $\mathrm{ran}(AB) = A(\mathrm{ran}\, B) \subset \mathrm{ran}\, A$, hence $\dim\mathrm{ran}(AB) \le \dim\mathrm{ran}\, B$ (since a linear map can only decrease the dimension) and $\dim\mathrm{ran}(AB) \le \dim\mathrm{ran}\, A$.

8.5 For any basis $V \in L(\mathbb{F}^n, X)$ of X, with dual basis Λ^{t}, $\mathrm{id}_X = V\Lambda^{\mathrm{t}}$ while $\Lambda^{\mathrm{t}}V = \mathrm{id}_n$, hence $\mathrm{trace}(\mathrm{id}_X) = \mathrm{trace}(\mathrm{id}_n) = n$.

8.6 We learned to write such P as $V\Lambda^{\mathrm{t}}$ with V a basis for $\mathrm{ran}\, P$ and $\Lambda^{\mathrm{t}}V = \mathrm{id}$, hence $\mathrm{trace}(P) = \mathrm{trace}(\Lambda^{\mathrm{t}}V) = \#V = \dim\mathrm{ran}\, P$.

8.7 Since the collection of all rank-1 linear maps on our finite-dimensional vector space A is spanning for $L(X)$, any two linear maps on $L(X)$ that agree on that collection agree on all of $L(X)$. Hence only need to show that trace is linear. With W a fixed basis for X, the map $A \mapsto W^{-1}AW$ is linear, as is the map $\mathbb{F}^{n\times n} \to \mathbb{F} : \hat{A} \mapsto \mathrm{trace}(\hat{A})$. This proves that trace is a linear map.

8.8 With V and W bases for X and Y, respectively,

$$\mathrm{trace}(AB) = \mathrm{trace}(WAV^{-1}VBW^{-1}) = \mathrm{trace}(VBW^{-1}WAV^{-1}) = \mathrm{trace}\, BA.$$

8.9 It is sufficient to restrict attention to the subspace $\mathrm{ran}\, V + \mathrm{ran}\, W$, and it is finite-dimensional, hence the earlier result provides the implication here, too.

8.10 With $x =: x_1 + \mathrm{i}x_2$, $zx = x_1a - x_2b + \mathrm{i}(x_1b + x_2a)$, hence the matrix is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

8.11 (a) F (e.g., $M = 0 = B$); (b) T (since then also $A = BM^{-1}$ hence $\mathrm{ran}\, A = \mathrm{ran}\, B$); (c) T (since then the same columns are bound in A and in B).