Answers to all problems in chapter 8

8.1 The rrref for this matrix is

\[
\text{rrref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix}
\]

hence \( A = A(:, [1, 2]) \text{rrref}(A) \) with \( A(:, [1, 2]) \) a basis for \( \text{ran} A \), therefore this factorization is minimal.

8.2 From the minimal factorization in previous problem, \( A(:, 1:2) \) is a basis for \( \text{ran} A \), and \( \text{rrref}(A)^t \) is a basis for \( A^t \).

8.3 \( A = [e_1, e_2, 0, 0], \ B = [0, 0, e_3, e_4] \)

8.4 Let \( A = V_A \Lambda A \Bbbk B = V_B \lambda_B \) be minimal factorizations, therefore \( \text{rank} A = \# V_A, \ \text{rank} B = \# V_B \).

Then \( AB = V_A (\Lambda_A V_B \Lambda_B) \) which implies that \( \text{rank}(AB) \leq \# \) \( V_A = \text{rank} A \). Also \( AB = (V_A \Lambda A V_B) \Lambda_B \) which implies that \( \text{rank}(AB) \leq \# (V_A \Lambda_A V_B) = \# V_B = \text{rank} B \). Therefore, \( \text{rank}(AB) \leq \text{min} \{\text{rank} A, \text{rank} B\} \).

The alternative argument: \( \text{ran} AB \subseteq \text{ran} A \) implies that \( \text{rank} (AB) \leq \text{rank} A \), while also \( \text{ran} (AB) = A(\text{ran} B) = \text{ran} (A|_{\text{ran} B}) \), hence \( \text{rank} AB = \dim \text{ran}(AB) = \dim \text{ran} (A|_{\text{ran} B}) \leq \dim \text{dom}(A|_{\text{ran} B}) = \dim \text{ran} B = \text{rank} B \), the inequality by the Dimension Formula.

The most direct argument: \( \text{ran}(AB) = A(\text{ran} B) \subseteq \text{ran} A \), hence \( \dim \text{ran}(AB) \leq \dim \text{ran} B \) (since a linear map can only decrease the dimension) and \( \dim \text{ran}(AB) \leq \dim \text{ran} A \).

8.5 For any basis \( V \in L(\mathbb{F}^n, X) \) of \( X \), with dual basis \( \Lambda^t \), \( \text{id}_X = VA^t \) while \( \Lambda^t V = \text{id}_n \), hence \( \text{trace}(\text{id}_X) = \text{trace}(\text{id}_n) = n \).

8.6 We learned to write such \( P \) as \( VA^t \) with \( V \) a basis for \( \text{ran} P \) and \( \Lambda^t V = \text{id}_n \), hence \( \text{trace}(P) = \text{trace}(\Lambda^t V) = \# V = \dim \text{ran} P \).

8.7 Since the collection of all rank-1 linear maps on our finite-dimensional vector space \( A \) is spanning for \( L(X) \), any two linear maps on \( L(X) \) that agree on that collection agree on all of \( L(X) \). Hence only need to show that trace is linear. With \( W \) a fixed basis for \( X \), the map \( A \mapsto W^{-1} AW \) is linear, as is the map \( \mathbb{F}^{n \times n} \to \mathbb{F}: \ A \mapsto \text{trace}(A) \). This proves that trace is a linear map.

8.8 With \( V \) and \( W \) bases for \( X \) and \( Y \), respectively,

\[
\text{trace}(AB) = \text{trace}(WAV^{-1}VBW^{-1}) = \text{trace}(VBW^{-1}WAV^{-1}) = \text{trace}(BA)
\]

8.9 It is sufficient to restrict attention to the subspace \( \text{ran} V + \text{ran} W \), and it is finite-dimensional, hence the earlier result provides the implication here, too.

8.10 With \( x =: x_1 + ix_2, \ 2x = x_1 a - x_2 b + i(x_1 b + x_2 a) \), hence the matrix is \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \).

8.11 (a) F (e.g., \( M = 0 = B \)); (b) T (since then also \( A = BM^{-1} \) hence \( \text{ran} A = \text{ran} B \)); (c) T (since then the same columns are bound in \( A \) and in \( B \)).