Answers to all problems in chapter 8

8.1 The rrref for this matrix is

$$\text{rrref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix}$$

hence $A = A(:, [1,2])\text{rrref}(A)$ with $A(:, [1,2])$ a basis for $\operatorname{ran} A$, therefore this factorization is minimal.

8.2 From the minimal factorization in previous problem, $A(:, 1{:}2)$ is a basis for $\operatorname{ran} A$, and $\text{rrref}(A)^{\mathrm{t}}$ is a basis for A^{t}.

8.3 $A = [e_1, e_2, 0, 0]$, $B = [0, 0, e_3, e_4]$

8.4 Let $A = V_A \Lambda_A$ and $B = V_B \lambda_B$ be minimal factorizations, therefore $\operatorname{rank} A = \#V_A$, $\operatorname{rank} B = \#V_B$. Then $AB = V_A(\Lambda_A V_B \Lambda_B)$ which implies that $\operatorname{rank}(AB) \le \#V_A = \operatorname{rank} A$. Also $AB = (V_A \Lambda_A V_B)\Lambda_B$ which implies that $\operatorname{rank}(AB) \le \#(V_A \Lambda_A V_B) = \#V_B = \operatorname{rank} B$. Therefore, $\operatorname{rank}(AB) \le \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

The alternative argument: $\operatorname{ran} AB \subseteq \operatorname{ran} A$ implies that $\operatorname{rank}(AB) \le \operatorname{rank} A$, while also $\operatorname{ran}(AB) = A(\operatorname{ran} B) = \operatorname{ran}(A_{|\operatorname{ran} B})$, hence $\operatorname{rank} AB = \dim \operatorname{ran}(AB) = \dim \operatorname{ran}(A_{|\operatorname{ran} B}) \le \dim \operatorname{dom}(A_{|\operatorname{ran} B}) = \dim \operatorname{ran} B = \operatorname{rank} B$, the inequality by the Dimension Formula.

The most direct argument: $\operatorname{ran}(AB) = A(\operatorname{ran} B) \subset \operatorname{ran} A$, hence $\dim \operatorname{ran}(AB) \le \dim \operatorname{ran} B$ (since a linear map can only decrease the dimension) and $\dim \operatorname{ran}(AB) \le \dim \operatorname{ran} A$.

8.5 For any basis $V \in L(\mathbb{F}^n, X)$ of X, with dual basis Λ^{t}, $\mathrm{id}_X = V\Lambda^{\mathrm{t}}$ while $\Lambda^{\mathrm{t}} V = \mathrm{id}_n$, hence $\operatorname{trace}(\mathrm{id}_X) = \operatorname{trace}(\mathrm{id}_n) = n$.

8.6 We learned to write such P as $V\Lambda^{\mathrm{t}}$ with V a basis for $\operatorname{ran} P$ and $\Lambda^{\mathrm{t}} V = \mathrm{id}$, hence $\operatorname{trace}(P) = \operatorname{trace}(\Lambda^{\mathrm{t}} V) = \#V = \dim \operatorname{ran} P$.

8.7 Since the collection of all rank-1 linear maps on our finite-dimensional vector space A is spanning for $L(X)$, any two linear maps on $L(X)$ that agree on that collection agree on all of $L(X)$. Hence only need to show that trace is linear. With W a fixed basis for X, the map $A \mapsto W^{-1}AW$ is linear, as is the map $\mathbb{F}^{n\times n} \to \mathbb{F} : \hat{A} \mapsto \operatorname{trace}(\hat{A})$. This proves that trace is a linear map.

8.8 With V and W bases for X and Y, respectively,

$$\operatorname{trace}(AB) = \operatorname{trace}(WAV^{-1}VBW^{-1}) = \operatorname{trace}(VBW^{-1}WAV^{-1}) = \operatorname{trace} BA.$$

8.9 It is sufficient to restrict attention to the subspace $\operatorname{ran} V + \operatorname{ran} W$, and it is finite-dimensional, hence the earlier result provides the implication here, too.

8.10 With $x =: x_1 + \mathrm{i}x_2$, $zx = x_1 a - x_2 b + \mathrm{i}(x_1 b + x_2 a)$, hence the matrix is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

8.11 (a) F (e.g., $M = 0 = B$); (b) T (since then also $A = BM^{-1}$ hence $\operatorname{ran} A = \operatorname{ran} B$); (c) T (since then the same columns are bound in A and in B).