more practice problems, taken from the book by Daniels and Noble =: [DN]. (actually, book provides answers to many of its problems, and that would be a good source of problems for you, if you can get hold of a copy; there should be a copy on reserve in the math library)

1. without doing any calculation, determine `bounds` and `frees` for the following matrices from page 141 of [DN] (which, at that point, complains about the fact that these are not in reduced row echelon form (called Gauss-reduced form in the book):

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

2. determine `bounds` and `frees` for the matrix $[\mathrm{id}_n, B]$, with $B \in \mathbb{F}^{n \times m}$ arbitrary.

3. problem 4.2.10: if $A \in \mathbb{F}^{p \times q}$ with exactly $p - n$ rows identically zero, and if you know that $\dim \operatorname{ran} A = k$, show that then $k \le n$.

4. somewhat problem 4.2.17: show that the rrref of an invertible matrix is the identity matrix.

5. problem 4.3.3.

6. a simple version of 4.3.15: let $t_1 < t_2$. Prove that, for arbitrary $a \in \mathbb{R}^4$, there exists exactly one cubic polynomial p for which

$$(p(t_1), p'(t_1), p(t_2), p'(t_2)) = a.$$

7. problem 4.3.8

1. (answers: `bounds` = (1,2), `frees` =(); `bounds` = (1,2), `frees` = (3); `bounds` = (1,2,3), `frees` = () (I cheated here, by saying to myself that there is a pivot row for each unknown if I choose to eliminate from the right).)

2. (answer: the matrix is already in rref since its first columns form an identity matrix, so `bounds` $= 1{:}n$, `frees` $= n+1{:}n+m$.)

3. (answer: by assumption, at most n rows can be used as pivot rows, hence at most n columns can be bound. Since we know that $A(:,\texttt{bounds})$ is a basis for $\operatorname{ran} A$, it follows that $k = \dim \operatorname{ran} A = \#\texttt{bounds} \le n$.)

4. (answer: A invertible implies that A is square and that all its columns are bound. Hence its rref must be square, too, and each of its rows must be a pivot row, hence, it must be equal to the rrref, and since the bound part of the latter must be an identity matrix, yet its bound part is everything, it must be an identity matrix.)

5. (answer: we are, in effect, asked to describe $\operatorname{ran} A$. Using row 2 as pivot row for first unknown, the remaining rows become

$$\begin{bmatrix} 0 & 19 & -2 & -6 \\ 0 & 19 & -2 & -6 \end{bmatrix}$$

Hence also unknown 2 is bound, but the other unknowns are free. Hence $A(:,(1:2))$ is a basis for $\operatorname{ran} A$. (The book's answer in the back goes further, by pointing out, in effect, a different basis for $\operatorname{ran} A$, namely

$$B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and, certainly, you can verify that the first two columns of A are in the range of this matrix (i.e., satisfy that $x_3 = x_1 + x_2$), but the book doesn't tell me how they came by their answer (I believe they used the fact that $\operatorname{ran} A$ is the orthogonal complement of the nullspace of A^{t}.))

6. (answer: Let $Q : \Pi_3 \to \mathbb{R}^4 : p \mapsto (p(t_1), p'(t_1), p(t_2), p'(t_2))$. If $Qp = 0$, then p has a double zero at t_1 and at t_2, hence must be of the form $p(s) = q(s)(s-t_1)^2(s-t_2)^2$ for some polynomial q. Therefore, since we also know that p is a cubic polynomial, then necessarily $q = 0$, hence $p = 0$. In other words, Q is 1-1. Since we know that $\dim \Pi_3 = 4 = \dim \mathbb{R}^4 = \dim \operatorname{tar} Q$, it follows that Q must also be onto, hence invertible. But that says exactly that, for every $a \in \mathbb{R}^4$, there is exactly one $p \in \Pi_3 = \operatorname{dom} Q$ with $Qp = a$.)

7. (answer: (a) since we are after $\operatorname{null} A$ here, here is an amusing variant: pick row 1 as pivot row for x_1, getting the following changed row

$$\begin{bmatrix} 0 & -5 & 1 \end{bmatrix}.$$

That 1 in column 3 looks so enticing, I'll use this row as pivot row for unknown x_3, eliminating it from row 1, to get altogether the matrix

$$R := \begin{bmatrix} 1 & 7 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

with the same nullspace as A and with $R(:\ [1,3])$ the identity. Therefore, the theory developed in class tells us that the matrix

$$C = \begin{bmatrix} -7 \\ 1 \\ 5 \end{bmatrix}$$

is a basis for $\operatorname{null} A$.

(b) This makes the general solution of $A? = (3,0)$ the vector $(2-7t, 1+t, -1+5t)$.)