443 Fall 02 practice problems 1

This is a list of additional problems you might want to try. Don't hand them in. However, if working on them raises some questions, by all means raise them with me.

1. Give as simple an example as you can of a set X and two maps f and g, both from X to X, for which $fg \neq gf$.

2. Give the simplest example you can of maps f, g, h, for which fg and fh are defined and equal, yet $g \neq h$.

3. Give the simplest example you can of maps f, g, h, for which fh and gh are defined and equal, yet $f \neq g$.

4. Which of the following maps is the same as the map

$$f : [0\,..\,1] \to [0\,..\,1] : t \mapsto \sin(t) \quad ?$$

(a) $[0\,..\,1] \to [0\,..\,1] : t \mapsto \sqrt{1-\cos(t)^2}$;
(b) $[0\,..\,1] \to [0\,..\,(.5)] : t \mapsto \sin(t)$;
(c) sin;
(d) $[0\,..\,1] \to [0\,..\,1] : t \mapsto \frac{\sin(t+q)+\sin(t-q)}{2\cos(q)}$, with q some fixed nonzero real.

5. How many maps are there from $\underline{3}$ to $\underline{3}$? How many of these are also onto? How many of the latter are also 1-1?

6. Prove that any map can have at most one inverse.

7. Prove that, for any invertible map f, f^{-1} is invertible.

8. For the map $f : \underline{6} \to \underline{6}$ given by the list $(3, 2, 5, 4, 2, 3)$, determine $f^{-1}Z$ for each of the following sets Z: (a) $\{1, 2, 3\}$; (b) $\{4, 5, 6\}$; (c) $\{2, 2\}$; (d) $\{2\}$; (e) $\{6, 1\}$.

Also, for each $b \in \operatorname{tar} f$, determine all solutions of the equation $f(?) = b$.

What is the minimum number of values of f that would have to be changed in order to get an invertible map?

9. Prove that if $fg = \mathrm{id}$, then gf is idempotent, i.e., is the identity on its range, and that its range equals $\operatorname{ran} g$.

10. Let $f : X \to Y$ and $g : Y \to Z$.
(a) Prove: if both f and g are invertible, then so is their composition, gf, and the inverse is $(gf)^{-1} = f^{-1}g^{-1}$.
(b) Give an example to show that, if gf is invertible, then neither f nor g need to be invertible.
(c) Assuming that all three sets are finite, what follows from the invertibility of gf about $\#X$, $\#Y$, and $\#Z$?

(d) Assuming that gf is invertible and all three sets are finite, what is the weakest assumption on $\#Y$ which would guarantee invertibility of both f and g?

11. For each of the following maps, state whether or not it is linear. If your answer is in the negative, give a reason.

(a) $m_g : C[0\,..\,1] \to C[0\,..\,1] : f \mapsto g.f$, with g a fixed element of $C[0\,..\,1]$ and

$$g.f : [0\,..\,1] \to \mathbb{F} : t \mapsto g(t)f(t)$$

the **pointwise product** of the two functions, g and f.

(b) $\sigma_h : C(\mathbb{R}) \to C(\mathbb{R}) : f \mapsto f(h\cdot)$, with

$$f(h\cdot) : \mathbb{R} \to \mathbb{R} : t \mapsto f(ht)$$

the h**-dilate** of f.

(c) : $\tau_h : C(\mathbb{R}) \to C(\mathbb{R}) : f \mapsto f(h+\cdot)$, with

$$f(h+\cdot) : \mathbb{R} \to \mathbb{R} : t \mapsto f(h+t)$$

the h**-translate** of f.

(d) : $C(\mathbb{R}) \to C(\mathbb{R}) : f \mapsto f \circ g$, with g a fixed element of $C(\mathbb{R})$.

(e) : $C(\mathbb{R}) \to C(\mathbb{R}) : f \mapsto g \circ f$, with g a fixed element of $C(\mathbb{R})\backslash\Pi_1$.

(f) : $C(\mathbb{R}) \to C(\mathbb{R}) : f \mapsto g + f$, with g a fixed element of $C(\mathbb{R})\backslash 0$.

(g) : $\mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n} : A \mapsto (A + A^{\mathrm{c}})/2$.

(h) : $\mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n} : A \mapsto (A + A^{\mathrm{t}})/2$.

(i) : $\mathbb{F}^n \to \mathbb{F}^m : x \mapsto (\sum_{j=1}^n A(i,j)x_j : i = 1,\ldots,m)$, with A a fixed element of $\mathbb{F}^{m\times n}$.

(j) : $\mathbb{F}^{n\times n} \to \mathbb{F} : A \mapsto \operatorname{trace} A := \sum_{j=1}^n A(j,j)$

(k) : $\mathbb{F}^{m\times n} \to \mathbb{F}^{m\times n} : A \mapsto B.A := (B(i,j)A(i,j) : i = 1,\ldots,m; j = 1,\ldots,n)$, with B a fixed element of $\mathbb{F}^{m\times n}$.

(l) : $\mathbb{R}^n \to \mathbb{R} : a \mapsto \max(a)$.

(m) : $C^{(2)}[0\,..\,1] \to C[0\,..\,1] : f \mapsto g.f - h.(Df) + D^2 f$, with g, h fixed elements of $C[0\,..\,1]$, and $D^j f$ the jth derivative of f.

(n) : $\mathbb{R}^3 \to \mathbb{R}^3 : a \mapsto (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$, with b a fixed element of $\mathbb{R}^3$.

(o) : $\mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}$.

(p) : $C[0\,..\,1] \to C[0\,..\,1] : f \mapsto \int_0^1 k(\cdot,t)f(t)\,\mathrm{d}t$, with k a fixed element of $C([0\,..\,1]^2)$.

Answers

If you don't agree in any way, do email me.

1. $X = \{1, 2\}$, $f = (1, 1)$, $g = (2, 2)$.

2. $X = \{1, 2\}$, $f = (1, 1)$, $g = (2, 2)$, $h = (1, 2)$.

3. $X = \{1, 2\}$, $f = (2, 1)$, $g = (1, 1)$, $h = (2, 2)$.

4. (a), (d)

5. $\#\underline{3}^{\underline{3}} = 3^3$; there are $3! = 6$ onto maps and, by pigeonhole principle, each must also be 1-1.

6. if $f : T \to U$ and $g, h : U \to T$ both are inverses of f, then $g = g\mathrm{id} = g(fh) = (gf)h = \mathrm{id}h = h$. Note that this only uses that g is a left inverse and h is a right inverse.

7. if f is invertible, then $ff^{-1} = \mathrm{id}$ and $f^{-1}f = \mathrm{id}$, hence f is both a left and a right inverse for inf f, hence must be its inverse.

8. (a) $\{1, 2, 5, 6\}$; (b) $\{3, 4\}$; (c) $\{2, 5\}$; (d) $\{2, 5\}$; (e) $\{\}$. Also, $f^{-1}\{1\} = \{\} = f^{-1}\{6\}$; $f^{-1}\{2\} = \{2, 5\}$; $f^{-1}\{3\} = \{1, 6\}$; $f^{-1}\{4\} = \{4\}$; $f^{-1}\{5\} = \{3\}$; Also, changing just $f(1)$ and $f(2)$ (to 1 and 6, respectively), would make f 1-1, hence onto, hence invertible.

9. $(gf)(gf) = g(fg)f = gf$; if $t = g(s)$ then $(gf)(t) = (gf)(g(s)) = g(fg)(s) = g(id)(s) = g(s) = t$, hence $\operatorname{ran} g \subset \{t \in \operatorname{dom} f : gf(t) = t\} \subset \operatorname{ran} gf \subset \operatorname{ran} g$, therefore all these sets must be equal.

10. (a) $(f^{-1}g^{-1})(gf) = f^{-1}(g^{-1}g)f = f^{-1}f = \mathrm{id}$. Similarly $(gf)(f^{-1}g^{-1}) = \mathrm{id}$.
(b) $f = (1)$, $g = (1, 1)$, with $X = \{1\} = Z$, $Y = \{1, 2\}$.
(c) $\#X = \#Y = \#Z$.
(d) $\#Y \le \#X$.

11. (a) T; (b) T; (c) T; (d) T; (e) F (since $g \notin \Pi_1$, can find σ and τ with $g(\sigma + \tau) \neq g(\sigma) + g(\tau)$, hence $g \circ (f + h) \neq g \circ f + g \circ h$ for the constant functions $f : t \mapsto \sigma$ and $h : t \mapsto \tau$); (f) F (since $0 + g \neq 0$); (g) T; (h) T; (i) T!!!; (j) T (after correction); (k) T (like (a)); (l) F ($\max(-\mathbf{i}_1) = 0 \neq -1 = -\max(\mathbf{i}_1)$); (m) T; (n) T (well, after correction); (o) T; (p) T.