Euclid’s Algorithm

Horner’s method is a special case of Euclid’s Algorithm which constructs, for given polynomials \( p \) and \( h \neq 0 \), (unique) polynomials \( q \) and \( r \) with \( \deg r < \deg h \) so that

\[
p = hq + r.
\]

For variety, here is a nonstandard discussion of this algorithm, in terms of elimination.

Assume that

\[
h(t) = a_0 + a_1 t + \cdots + a_d t^d, \quad a_d \neq 0,
\]

and

\[
p(t) = b_0 + b_1 t + \cdots + b_n t^n.
\]

Then we seek a polynomial

\[
q(t) = c_0 + c_1 t + \cdots + c_{n-d} t^{n-d}
\]

for which

\[
r := p - hq
\]

has degree \( < d \). This amounts to the square upper triangular linear system

\[
\begin{align*}
a_d c_0 + a_{d-1} c_1 + \cdots + a_0 c_d &= b_d \\
a_d c_1 + a_{d-1} c_2 + \cdots + a_0 c_{d+1} &= b_{d+1} \\
& \quad \vdots \\
a_d c_{n-d-1} + a_{d-1} c_{n-d} &= b_{n-1} \\
a_d c_{n-d} &= b_n
\end{align*}
\]

for the unknown coefficients \( c_0, \ldots, c_{n-d} \) which can be uniquely solved by back substitution since its diagonal entries all equal \( a_d \neq 0 \).
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