887 Spring '03 ANSWERS to assignment 1

(1) If $k := \sup_n \deg p_n < \infty$, then (p_n) is in Π_k, a finite-dimensional linear space, hence $\Pi_k|_{[a..b]}$ is closed, therefore $f \in \Pi_k|_{[a..b]}$. (The problem should have said that $f \notin \Pi|_{[a..b]}$.)

(2) The constant sequence $(U_n := U : n \in \mathbb{N})$ with $F := \{()^0, ()^1, ()^2\}$ satisfies all assumptions of Korovkin's theorem, hence converges pointwise to the identity on $X := C([a\,..\,b])$, hence $Uf = f$ for all $f \in X$.

(3) Anticipating the next problem, assume that f is continuous, with $f(0), f(1) \in \mathbb{Z}$. Set $B_n^{\lfloor\rfloor} f - B_n f =: \sum c(j)\gamma_{j,n-j}$. Then, for any $t \in [0\,..\,1]$,

$$|(B_n^{\lfloor\rfloor} f - B_n f)(t)| \le \max_j \left(|c(j)|/\binom{n}{j}\right) \sum_j \binom{n}{j} t^j (1-t)^{n-j} = \max_j |c(j)|/\binom{n}{j}.$$

However, $c(0) = 0 = c(n)$ while, in general, $|c(j)| < 1$. Since $\binom{n}{j} \ge n$ for $0 < j < n$, it follows that $\|B_n^{\lfloor\rfloor} f - B_n f\| < 1/n$, hence

$$\lim_n B_n^{\lfloor\rfloor} f = \lim B_n f = f.$$

(4) Sufficiency was already proved in previous problem. For necessity, note that, for any polynomial p with integer coefficients, $p(0)$ and $p(1)$ are necessarily integers, hence any uniform limit of such polynomials must have integer values at 0 and 1.