887 Spring '03 ANSWERS to assignment 2

(1) Let $f := ()^2$. By the notes, $M := \Pi_{0,2} = \cup_{\zeta\in[-1..1]} M_\zeta$, with $M_\zeta := \Pi_{0,(-1,\zeta,1)}$ a linear space, hence provides a unique best L_2-approximation, m_ζ, to f, and $\|f\|^2 = \|f-m_\zeta\|^2+\|m_\zeta\|^2$. Thus, minimizing $\|f-m_\zeta\|$ over ζ is the same as maximizing $\|m_\zeta\|$ over ζ. For $-1<\zeta<1$, the space M_ζ is two-dimensional, and the two functions $m_- := \chi_{[-1..\zeta]}$ and $m_+ := \chi_{[\zeta..1]}$, form an orthogonal basis for it. Hence $m_\zeta = \langle f, m_-\rangle/\langle m_-, m_-\rangle\, m_- + \langle f, m_+\rangle/\langle m_+, m_+\rangle\, m_+$. This formula even works for $\zeta = \pm 1$ if we agree that 0 times anything is zero.

Note that, with $g(\zeta) := \langle f, m_+\rangle/\langle m_+, m_+\rangle$, we have $g(-\zeta) := \langle f, m_-\rangle/\langle m_-, m_-\rangle$ (since this change of variable changes m_+ to m_-, yet leaves our f unchanged). Therefore,

$$\|m_\zeta\|^2 = g(\zeta)^2(1-\zeta) + g(-\zeta)^2(\zeta-(-1)),$$

and the second term is obtained from the first term by the substitution $\zeta \to -\zeta$. Hence, maximizing $\|m_\zeta\|^2$ over ζ is the same as maximizing the polynomial obtained from $\zeta \mapsto g(\zeta)^2(1-\zeta)$ by retaining only the even terms and dropping any positive common factor const.

We compute: $g(\zeta) = \int_\zeta^1 ()^2/(1-\zeta) = (1/3)(1-\zeta^3)/(1-\zeta)$. Hence

$$g(\zeta)^2(1-\zeta) = \text{const}(1-\zeta^3)^2/(1-\zeta) = \text{const}(1+\zeta+\zeta^2)(1-\zeta^3) = \text{const}(1+\zeta^2-\zeta^4+ \text{ odd terms}).$$

Thus, the sought-for optimal ζ are all the points in $[-1\,..\,1]$ at which

$$\zeta \mapsto 1+\zeta^2-\zeta^4$$

takes on its maximum (on that interval). By differentiation, the critical points solve the equation $2\zeta - 4\zeta^3 = 0$, with $\zeta = 0$ obviously a local minimum, while the function takes the same value, 1, also at the endpoints. This implies that $\zeta = \pm 1/\sqrt{2}$ are the maxima, and there are no others. Also, $g(\pm 1/\sqrt{2}) = (1/3)(1 \pm 1/\sqrt{2} + 1/2) = (3 \pm \sqrt{2})/6$ are the two heights of the corresponding ba's.

(2) Since $|(f-f_j)(n)| = |(f-f_{j-1})(n)|$ for all $n \neq j$, we have $\|f-f_j\| < \|f-f_{j-1}\|$ only if $|1-f(j)| < |f(j)|$. On the other hand, since $\lim_n f(n) = 0$, we must have $|f(j)| < 1/2$ for all j greater than some j_0, hence must have $\|f-f_j\| \geq \|f-f_{j-1}\|$ for all $j > j_0$. This shows that $\operatorname{dist}(f, M) = \inf_{j\leq j_0} \|f-f_j\|$, and this inf, being over a finite set, is taken on. This shows that M is an existence set.

Also M is bounded (since all its elements are of norm ≤ 1). However, for any $f \in c_0$, we can choose j with $|f(j)| < 1$, and, with $g := 2(f_j - f_{j-1})$, we have $\|f-g\| \geq |f(j)-2| > 1 = \limsup_n \|f_n - g\|$ since $\|f_n - g\| = 1$ for $n \geq j$. This shows that no subsequence of (f_n) can come close to any $f \in c_0$ (let alone any $f \in M$).

(3) $\|g - \alpha()^0\|_\infty = \max\{|1-\alpha|, |-1-\alpha|\}$ and this is uniquely minimized by $\alpha = 0$. But, for all $U = \{u_1, u_2\} \subset (-1\,..\,1)$, $\lambda := w_1\delta_{u_1} + w_2\delta_{u_2} \perp \Pi_0$ must have $w_1 + w_2 = 0$, hence if also $\|\lambda\| = 1$, need $u_1 \neq u_2$ and $|w_1| + |w_2| = 1$, but then $\lambda g = w_1 u_1 + w_2 u_2 = \pm(u_1 - u_2)/2 < 1 = \|\lambda\|\|g\|$.

(4) Let $V \in L(\mathbb{R}^n, M)$ be a basis for M, and let $-\pi \le u_1 < \cdots < u_n < \pi$. Then $u(t) := (u_j(t) := (1-t)u_j + tu_{j+1} : j = 1{:}n)$ with $u_{n+1} := u_1 + 2\pi = u_1$ depends continuously on t, therefore also $F : t \mapsto \det Q_{u(t)}V$ is a continuous function, with $F(1) = \det Q_{u_2,\ldots,u_n,u_1}V = (-1)^{n-1}\det Q_{u_1,u_2,\ldots,u_n}V = (-1)^{n-1}F(0)$, hence if n were even, then $F(t) = 0$ for some t and, since $u(t)$ is strictly increasing, M would not be Haar.