887 Spring '03 ANSWERS to assignment 3

(1) Where does the proof of Bernstein's Theorem, 62, break down if we were to use to prove that the entire sequence (p_k) is Cauchy in $C^{(r)}(\mathbb{T})$?

We would end up needing that $\sum_{k=j+1}^{J} k^r E_{k-1}(f)$ goes to zero as $j, J \to \infty$ while the theorem's assumption only ensure that $\sum_{k=j+1}^{J} k^{r-1} E_{k-1}(f)$ goes to zero as $j, J \to \infty$.

(2) Construct all ba's to $\chi_{[0..\alpha]}$ from $\Pi_0 \subset L_1([0\,..\,1]$.

Set $f := \chi_{[0..\alpha]}$, and let $c_\beta := \beta()^0$ be the general element of Π_0. One computes

$$e(\beta) := \|f - c_\beta\|_1 = |\beta|(\alpha)_+ + |1-\beta|(1-\alpha)_+.$$

This shows e to be piecewise linear, with breaks at 0 and 1 only and with with negative slope for $\beta < 0$ and positive slope for $1 < \beta$. Consequently, e takes on its minimum value, at 0 or 1. At these points, it interpolates to the data α_+ and $(1-\alpha)_+$, respectively, hence its minimum value is $\min\{\alpha_+, (1-\alpha)_+\}$, therefore

$$\mathcal{P}_M(f) = \begin{cases} \{0\}, & \alpha < 1/2; \\ \{c_\beta : 0 \le \beta \le 1\}, & \alpha = 1/2; \\ \{1\}, & \alpha > 1/2; \end{cases}.$$

(3) Let $\|\cdot\| := \|\cdot\|_\infty([-1\,..\,1])$. Use the fact that a continuous linear functional takes on its norm on the error in a best approximation from the kernel of the linear functional to prove that

$$\sup\{Df(1)/\|f\| : f \in \Pi_n \backslash 0\} = n^2.$$

(Hint: Consider C_n, the **Chebyshev polynomial of degree** n, i.e.,

$$C_n(\cos(t)) = \cos(nt), \quad \forall\ t.$$

You may take for granted that this implicit definition does, indeed, describe a polynomial of degree $\le n$. Further hint: Use the Lagrange form of the interpolating polynomial from Π_n at some $(n+1)$-set $U \subset [-1\,..\,1]$ to represent $f \mapsto Df(1)$ as $\lambda_{U,w}$, with U the extrema of C_n in $[-1\,..\,1]$.)

Let U be any (ordered) $n+1$-subset of $[-1\,..\,1]$. Then, for any $p \in \Pi_n$,

$$p = \sum_{u \in U} p(u)\ell_u,$$

with

$$\ell_u := \prod_{v \in U \backslash \{u\}} (\cdot - v)/(u - v).$$

Therefore, for any $p \in \Pi_n$,

$$Dp(1) = \sum_{u \in U} p(u) D\ell_u(1) =: \lambda_{U,w} p. \qquad (*)$$

Each ℓ_u is a polynomial of degree n that vanishes at n of the $n+1$ points in U. Hence, by Rolle's theorem, its first derivative is a polynomial of degree $n-1$ that vanishes at $n-1$ distinct points in $(\min(U)\,..\max(U))$, hence must be of one (nonzero) sign on $[\max(U)\,..\infty)$. In particular, $D\ell_u(1)$ is the same sign as the leading coefficient of ℓ_u, hence is positive or negative depending on whether the number of points in U strictly to the right of u is even or odd.

Now, directly from the definition of C_n, C_n has exactly $n+1$ extreme points, namely the points $u_k := \cos(k\pi/n)$, $k = 0, \ldots, n$. Moreover, $C_n(u_k) = \cos(k/pi) = (-1)^k$, hence positive or negative depending on whether the number of extreme points to the right of it is even or odd. Thus, for this choice for U, our linear functional $\lambda_{U,w}$ defined in (*) takes on its norm on C_n.

It follows that $\sup_{p \in \Pi_n} Dp(1)/\|p\| = DC_n(1)$. To compute the latter number, differentiate the defining equation for C_n to find that

$$DC_n(\cos t)(-\sin t) = n(-\sin nt),$$

hence $DC_n(\cos t) = n(\sin nt)/\sin t$ and, by L'Hopital's Rule, the right hand side goes to n^2 as $t \to 0$ (i.e., as $\cos t \to 1$).

(4) Let f be a bounded function, on $[-1\,..\,1]$, say, and set $E_{n,j}(f) := \inf\{\|f - p\|_\infty([-1\,..1]/n) : p \in \Pi_j\}$. Prove: $E_{n,0} = o(1/n)$ iff f is differentiable at 0 and $Df(0) = 0$.

$Df(0) = 0$ is equivalent to $\lim_{t\to 0} \delta_{t,0} = 0$, with $\delta_{t,0} = (f(t)-f(0))/(t-0)$. The latter condition implies that $\limsup_{t\to 0} |f(t)-f(0)|/|t| = 0$, i.e., $\lim_{t\downarrow}(1/t)\|f-f(0)\|_\infty([-t\,..t]) = 0$, therefore

$$E_{1/t,0}(f) \le \|f - f(0)\|_\infty[-t\,..\,t] = o(t).$$

This proves '$\Longleftarrow$'.

'$\Longrightarrow$': To prove: $\lim_{t\to 0} \delta_{t,0} = 0$. For $n = 1, 2, \ldots$, let $c_n \in \Pi_0$ be the uniform ba to f on $[-1/n\,..1/n]$. Then, for $1/(n+1) \le |t| \le 1/n$, $|f(t)-f(0)| = |(f(t)-c_n)-(c_n-f(0))| \le 2E_n(f,0) = o(1/n)$, while $|t-0| \ge (1/n)/2$, therefore $\lim_{t\to 0} \delta_{t,0} f = 0$.