(1) Prove that, with $a := t_0 < \ldots < t_k =: b$ and for $j = 1, \ldots, k-1$, $D^j M(\cdot|t_0, .., t_k)$ is orthogonal to $\Pi_{<j}$ with respect to the inner product

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx.$$

If $p \in \Pi_{<j}$, then j-fold integration by parts gives

$$\langle D^j M, p\rangle = \sum_{i<j} \pm D^{j-i} M D^i p|_a^b = 0$$

the first equality since $D^j p = 0$ and the second since $D^i M$ vanishes at a and b for all $i < k-1$. In particular, this holds even for $j = k-1$ since $D^{k-2}M$ is continuous pp, hence absolutely continuous.

(2) Prove that the B-spline is 'bell-shaped', i.e., that, for $j = 1, \ldots, k-1$ and with $t_0 < \ldots < t_k$, $D^j M(\cdot|t_0, .., t_k)$ has exactly j strong sign changes.

By differentiation formula, $D^j M \in \operatorname{span}(B_{i,k-j} : i = 0, \ldots, j)$, hence can have at most j strong sign changes. On the other hand, it must have at least that many since, if $x_1 < \cdots < x_r$ are all its sign changes in $[a \,..\, b]$, then, with $p := \prod_{i=1}^r (\cdot - x_i)$, the product $D^j M\, p$ is of one (positive or negative) sign on $[a \,..\, b]$ except for the finitely many points x_i, hence $\langle D^j M, p\rangle \neq 0$, therefore $r \geq j$, by problem 1.

(3) Prove that Schoenberg's variation-diminishing spline approximation,

$$Vf := \sum_i B_{i,k} f(t_i^*),$$

to f has order of approximation $|\mathbf{t}|^2$ but no better. Explicitly, prove that there is a constant $C = C_k$ so that, for all $f \in C^{(2)}[a \,..\, b]$ and for all knot sequences $\mathbf{t} = (t_1, \ldots, t_{n+k})$ in $[a \,..\, b]$ with $I_{k,\mathbf{t}} = [a \,..\, b]$,

$$\|f - Vf\| \leq C_k |\mathbf{t}|^2 \|D^2 f\|,$$

while, for some such f, $\|f - Vf\| \neq o(|\mathbf{t}|^2)$, with $\|\ \| := \|\ \|_\infty([a \,..\, b])$.

Look it up in 'A practical guide to splines'.

(4) The **Appell** polynomials for a given linear functional μ on $C(\mathbb{R})$ with $\mu()^0 = 1$ are, by definition, the polynomials $(p_j^\mu : j = 0, 1, 2, \ldots)$ with $p_j^\mu \in \Pi_j$, all j, and

$$\mu D^k p_j = \delta_{kj}.$$

(i) Prove that the definite article is justified, i.e., prove that, for each such μ, there is exactly one such polynomial sequence.

(ii) Prove that (therefore), if $\mu T = \mu$ for some $T : f \mapsto f(\alpha \cdot +\beta)$ with $\alpha \neq 0$, then $Tp_j^\mu = \alpha^j p_j^\mu$, all j.

(iii) With $\mu = (\delta_0 + \delta_1)/2$, prove that the function $E_n : \mathbb{R} \to \mathbb{R}$, which equals p_n^μ on $[0\,..\,1)$ and satisfies the functional equation

$$E(\cdot + 1) = -E,$$

is a spline of order $n+1$ with knot sequence $\mathbb{Z} = (\ldots, -2, -1, 0, 1, 2, \ldots)$, and is even (odd) with respect to the point $1/2$ if n is even (odd).

(i) The matrix $(\mu D^i ()^j : i, j = 0, 1, 2, \ldots)$ is triangular, with nonzero diagonal entries, hence invertible. The j-th column of its inverse provides, up to a scalar multiple, the power-form coefficients for p_j, and uniquely so.

(ii) Since $DT = \alpha TD$, have $\mu(D^i T p_j) = \alpha^i \mu T D^i p_j = \alpha^i \mu D^i P_J$, therefore $Tp_j = \alpha^j p_j$.

(iii) For $T : f \mapsto f(1 - \cdot)$ and the given μ, have $\mu T = T$ and $\alpha = -1$, hence $p_j(1 - \cdot) = (-1)^j p_j$, showing that p_j is odd (even) around $1/2$ for j odd (even).

With that, since E_j is pp of order $j+1$ with breaks at $\mathbb{Z}$, only need to prove that it is in C^{j-1} at each integer, and, since $E_j(\cdot + 1) = -E_j$, only need to verify this at 0 where, by that functional equation, this reduces to the assertion that $-D^r p_j(1) = D^r p_j(0)$ for $r < j$ which is exactly the condition $\mu D^r p_j = 0$ for $r < j$.

The symmetry around $1/2$ follows from that for p_j.