887 Spring '03 HOMEWORK assignment 3, due Monday, 10 Mar 03

(1) Where does the proof of Bernstein's Theorem, 62, break down if we were to use it to prove that the entire sequence (p_k) is Cauchy in $C^{(r)}(\mathbb{T})$?

(2) Construct all ba's to $\chi_{[0..\alpha]}$ from $\Pi_0 \subset L_1([0\,..\,1])$.

(3) Let $\|\cdot\| := \|\cdot\|_\infty([-1\,..\,1])$. Use the fact that a continuous linear functional takes on its norm on the error in a best approximation from the kernel of the linear functional to prove that

$$\sup\{Df(1)/\|f\| : f \in \Pi_n\backslash 0\} = n^2.$$

(Hint: Consider C_n, the **Chebyshev polynomial of degree** n, i.e.,

$$C_n(\cos(t)) = \cos(nt), \quad \forall\, t.$$

You may take for granted that this implicit definition does, indeed, describe a polynomial of degree $\le n$. Further hint: Use the Lagrange form of the interpolating polynomial from Π_n at some $(n+1)$-set $U \subset [-1\,..\,1]$ to represent $\lambda : \Pi_n \to \mathbb{R} : f \mapsto Df(1)$ as $\lambda_{U,w}$, with U the extrema of C_n in $[-1\,..\,1]$.)

(4) Let f be a bounded function, on $[-1\,..\,1]$, say, and set

$$E_{n,j}(f) := \inf\{\|f - p\|_\infty([-1\,..\,1]/n) : p \in \Pi_j\}.$$

Prove: $E_{n,0} = o(1/n)$ iff f is differentiable at 0 and $Df(0) = 0$.