887 Spring '03 HOMEWORK assignment 5, due Monday, 31 Mar 03

As with all homework, you are free to turn to the available literature for help, as long as you follow accepted scientific practice of acknowledging such help in your write-up.

(1) Prove that, with $a := t_0 < \cdots < t_k =: b$ and for $j = 1, \ldots, k-1$, $D^j M(\cdot|t_0, .., t_k)$ is orthogonal to $\Pi_{<j}$ with respect to the inner product

$$\langle f, g \rangle := \int_a^b f(x) g(x)\, dx.$$

(2) Prove that the B-spline is 'bell-shaped', i.e., that, for $j = 1, \ldots, k-1$ and with $t_0 < \ldots < t_k$, $D^j M(\cdot|t_0, .., t_k)$ has exactly j strong sign changes.

(3) Prove that Schoenberg's variation-diminishing spline approximation,

$$Vf := \sum_i B_{i,k} f(t_i^*),$$

to f has order of approximation $|\mathbf{t}|^2$ but no better. Explicitly, prove that there is a constant $C = C_k$ so that, for all $f \in C^{(2)}[a\,..\,b]$ and for all knot sequences $\mathbf{t} = (t_1, \ldots, t_{n+k})$ in $[a\,..\,b]$ with $I_{k,\mathbf{t}} = [a\,..\,b]$,

$$\|f - Vf\| \le C_k |\mathbf{t}|^2 \|D^2 f\|,$$

while, for some such f, $\|f - Vf\| \ne o(|\mathbf{t}|^2)$, with $\|\ \| := \|\ \|_\infty([a\,..\,b])$.

(4) The **Appell** polynomials for a given linear functional μ on $C(\mathbb{R})$ with $\mu()^0 = 1$ are, by definition, the polynomials $(p_j^\mu : j = 0, 1, 2, \ldots)$ with $p_j^\mu \in \Pi_j$, all j, and

$$\mu D^k p_j = \delta_{kj}.$$

(i) Prove that the definite article is justified, i.e., prove that, for each such μ, there is exactly one such polynomial sequence.

(ii) Prove that (therefore), if $\mu T = \mu$ for some $T : f \mapsto f(\alpha \cdot + \beta)$ with $\alpha \ne 0$, then $T p_j^\mu = \alpha^j p_j^\mu$, all j.

(iii) With $\mu = (\delta_0 + \delta_1)/2$, prove that the function $E_n : \mathbb{R} \to \mathbb{R}$, which equals p_n^μ on $[0\,..\,1)$ and satisfies the functional equation

$$E(\cdot + 1) = -E,$$

is a spline of order $n+1$ with knot sequence $\mathbb{Z} = (\ldots, -2, -1, 0, 1, 2, \ldots)$, and is even (odd) with respect to the point $1/2$ if n is even (odd).