(1) Prove that the splines $E_n$ of a previous homework satisfy the recurrence

$$E_{n+1} = \int_0^1 E_n(\cdot + t) \, dt / (-2).$$

For the record, the function

$$\mathcal{E}_n := E(\cdot - (n+1)/2)/E(- (n+1)/2)$$

is called the Euler spline of degree $n$. It has simple knots, at $(n/2) + \mathbb{Z}$, and satisfies $\mathcal{E}_n(m) = (-1)^m$, all $m \in \mathbb{Z}$.

(2) A monospline of degree $k$ with knot sequence $t$ is, by definition, any element of $\Pi_k + S_{k,t}$.

Show that monosplines with simple knots occur naturally as the Peano kernel in a quadrature rule based on function values.

(3) With $\mu := \int_0^1$, show that the 1-periodic function $B_k$ that agrees with $p^\mu_k$ on $[0..1)$ is a monospline of degree $k$. For the record, $B_k$ is the Bernoulli spline of degree $k$.

(4) Following Glaeser (and Schoenberg), a (polynomial) spline is called perfect if its highest nontrivial derivative is absolutely constant. For example, the Euler spline $\mathcal{E}_n$ of a previous homework is perfect.

Prove that $M(\cdot|t_0, \ldots, t_k)$ is perfect if $t_j = \cos(k - j)\pi/k$, all $j$. 